Karlsruhe Institute of Technology

High Performance Humanoid Technologies

Robotics I: Introduction to Robotics
Exercise 2: Kinematics

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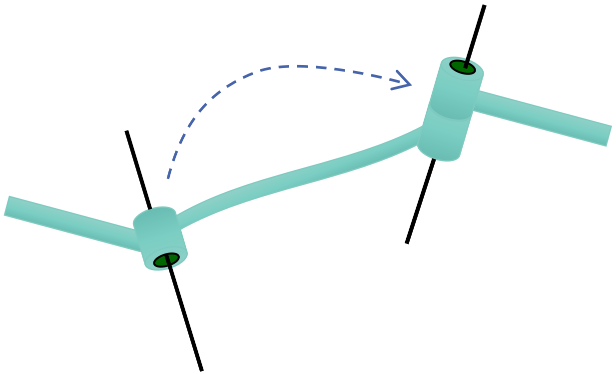
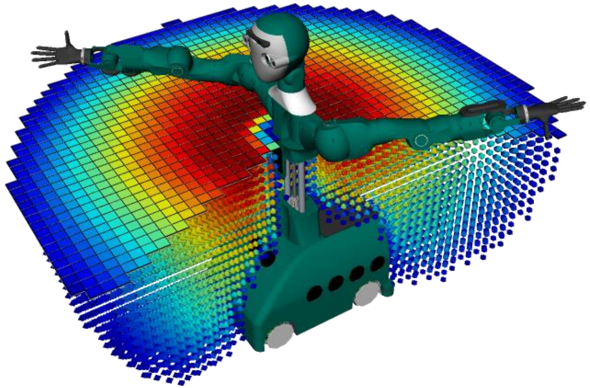
https://www.humanoids.kit.edu





Task 6 & 7 From Exercise 01

- Brief summary
  (see slides of previous exercise for full version)

Exercise 6: Quaternions

Show that the space of unit quaternions S^3 is a subgroup of the quaternions $\mathbb{H}$.

Remark: G is a group $(G, \ \cdot)$ if and only if:

1. Closed w.r.t. $(\cdot)$: $\forall a, b \in G : a \cdot b \in G$
2. Associativity: $\forall a, b, c \in G : (a \cdot b) \cdot c = a \cdot (b \cdot c)$
3. Identity element: $\exists e \in G : \forall a \in G : e \cdot a = a \cdot e = a$
4. Inverse element: $\forall a \in G : \exists a^{-1} : a \cdot a^{-1} = e$

Exercise 6.1: Closure (Tricky)

1. Closed w.r.t. $(\cdot)$: $\forall a, b \in G : a \cdot b \in G$

$$\forall \boldsymbol{a}, \boldsymbol{b} \in S^3 : \boldsymbol{a} \cdot \boldsymbol{b} \in S^3$$

$$\|\boldsymbol{a} \cdot \boldsymbol{b}\|^2 =$$

Exercise 6.1: Closure (Tricky)

1. Closed w.r.t. $(\cdot)$: $\forall a, b \in G : a \cdot b \in G$

$$\forall \boldsymbol{a}, \boldsymbol{b} \in S^3 : \boldsymbol{a} \cdot \boldsymbol{b} \in S^3$$

$$\|\boldsymbol{a} \cdot \boldsymbol{b}\|^2 = \|(a_0, a_1, a_2, a_3) \cdot (b_0, b_1, b_2, b_3)\|^2$$

$$\begin{aligned} &= a_3^2 b_3^2 + a_2^2 b_3^2 + a_1^2 b_3^2 + a_0^2 b_3^2 \\ &+ a_3^2 b_2^2 + a_2^2 b_2^2 + a_1^2 b_2^2 + a_0^2 b_2^2 \\ &+ a_3^2 b_1^2 + a_2^2 b_1^2 + a_1^2 b_1^2 + a_0^2 b_1^2 \\ &+ a_3^2 b_0^2 + a_2^2 b_0^2 + a_1^2 b_0^2 + a_0^2 b_0^2 \end{aligned}$$

Exercise 6.1: Closure (Tricky)

1. Closed w.r.t. $(\cdot)$: $\forall a, b \in G : a \cdot b \in G$

$$\forall \boldsymbol{a}, \boldsymbol{b} \in S^3 : \boldsymbol{a} \cdot \boldsymbol{b} \in S^3$$

$$\|\boldsymbol{a} \cdot \boldsymbol{b}\|^2 = \|(a_0, a_1, a_2, a_3) \cdot (b_0, b_1, b_2, b_3)\|^2$$

$$= b_3^2 \cdot \|\boldsymbol{a}\|^2 + b_2^2 \cdot \|\boldsymbol{a}\|^2 + b_1^2 \cdot \|\boldsymbol{a}\|^2 + b_0^2 \cdot \|\boldsymbol{a}\|^2$$

$$= (b_3^2 + b_2^2 + b_1^2 + b_0^2) \cdot \|\boldsymbol{a}\|^2$$

$$= \|\boldsymbol{b}\|^2 \cdot \|\boldsymbol{a}\|^2 = 1 \cdot 1 = 1$$

Exercise 6.2 & 6.3: Associativity and Identity Element

2. Associativity: $\forall a, b, c \in G : (a \cdot b) \cdot c = a \cdot (b \cdot c)$

3. Identity element: $\exists e \in G : \forall a \in G : e \cdot a = a \cdot e = a$

Exercise 6.2 & 6.3: Associativity and Identity Element

2. Associativity: $\forall a, b, c \in G : (a \cdot b) \cdot c = a \cdot (b \cdot c)$

Unit quaternions are a subset of quaternions. Multiplications of quaternions are associative.

3. Identity element: $\exists e \in G : \forall a \in G : e \cdot a = a \cdot e = a$

The identity element is $\boldsymbol{e} = (1, 0, 0, 0)$.

Exercise 6.4: Inverse Element

4. Inverse element: $\forall a \in G : \exists a^{-1} : a \cdot a^{-1} = e$

$$\boldsymbol{q} \in S^3 \Rightarrow \boldsymbol{q}^{-1} \in S^3$$

Exercise 6.4: Inverse Element

4. Inverse element: $\forall a \in G : \exists a^{-1} : a \cdot a^{-1} = e$

$$\boldsymbol{q} \in S^3 \Rightarrow \boldsymbol{q}^{-1} \in S^3$$

$$\|\boldsymbol{q}^{-1}\|^2 = \left\|\frac{\boldsymbol{q}^*}{\|\boldsymbol{q}\|^2}\right\|^2$$

$$= \left\|\frac{\boldsymbol{q}^*}{1}\right\|^2$$

$$= 1$$

Exercise 7: Rotations and Machine Learning

Rotations as input and output of learned models

1. Compare the **representations of rotations** as
   - Euler angles,
   - Quaternions, and
   - Rotation matrices

   with respect to how suitable they are as the **output** of a machine learning approach (e.g., neural networks)

2. A neural network, which has been trained to output rotation matrices, yields the matrix A:

$$A = \begin{pmatrix} 0.6 & 0.1 & 0.1 \\ 0.5 & 0.9 & 0.5 \\ 0.1 & 0.0 & 0.7 \end{pmatrix}$$

   Determine a rotation matrix R that is as "close" to A as possible.

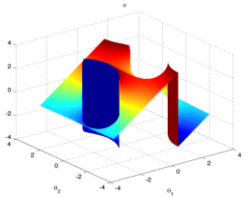
Exercise 7.1: Euler Angles and ML

Euler angles: $\alpha, \beta, \gamma \in [0, 2\pi]$

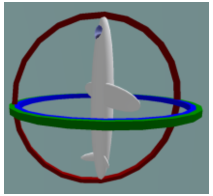
Rotation around three axes (several conventions)

- (+) Minimal representation for 3 Degree of Freedom
- (+) All values are valid, even beyond the intervall $[0, 2\pi]$

- (−) Not continuous



- (−) Multi-coverage
  A rotation can be describes by multiple tupels α, β, γ
  (e.g., Gimbal lock)



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Exercise 7.1: Quaternions and ML

$$S^3 = \{\, \mathbf{q} \in \mathbb{H} \mid \|\mathbf{q}\|^2 = 1 \,\}$$

$$\mathbf{q} = \left(\cos\frac{\Phi}{2}, \qquad \mathbf{a} \cdot \sin\frac{\Phi}{2}\right)$$

- ⊕ Easy to normalize $\quad \mathbf{p} \in \mathbb{H},\ \mathbf{p} \notin S^3: \quad \mathbf{q} = \frac{\mathbf{p}}{\|\mathbf{p}\|} \in S^3$
- ⊕ Local interpolation is linear:

$$\mathrm{SLERP}(\mathbf{q}_1, \mathbf{q}_2, t) = \frac{\sin(1-t)\,\theta}{\sin\theta} \cdot \mathbf{q}_1 + \frac{\sin t\theta}{\sin\theta} \cdot \mathbf{q}_2$$

- ○ Representation is not minimal: 1 redundant value ($\|\mathbf{q}\|^2 = 1$)
- ○ Double coverage: Each rotation can be described by two different unit quaternions

Exercise 7.1: Rotation Matrices and ML

$$R_{z,\alpha} = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- (+) Single coverage:
  A rotation corresponds to exactly one rotation matrix
- (+) Local interpolation is linear: $\sin\alpha \approx \alpha, \ \cos\alpha \approx 1, \ \text{for very small } \alpha$

- (○) Normalization is possible, but complex
  ➔ Gram-Schmidt, QR decomposition, SVD

- (−) Highly redundant representation: 6 redundant values

Exercise 7.2: Rotation Matrices and ML

A neural network, which has been trained to output rotation matrices, yields the matrix A:

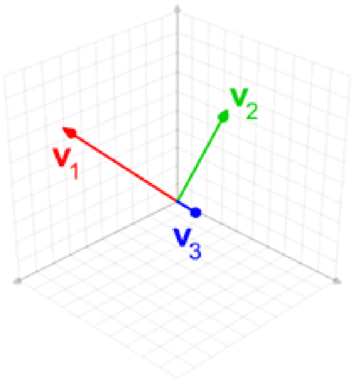
$$A = \begin{pmatrix} 0.6 & 0.1 & 0.1 \\ 0.5 & 0.9 & 0.5 \\ 0.1 & 0.0 & 0.7 \end{pmatrix}$$

Determine a rotation matrix R that is as “close” to A as possible.

Exercise 7.2: Gram-Schmidt Orthogonalization

- Orthogonalization
  - Given: Linearly independent vectors $w_1, \dots, w_3$
  - Unknown: Pairwise orthogonal vectors $v_1, \dots, v_3$ that span the same subspace

- Gram-Schmidt
  - $v_1 = w_1$
  - $v_2 = w_2 - \frac{v_1 \cdot w_2}{v_1 \cdot v_1} \cdot v_1$
  - $v_3 = w_3 - \frac{v_1 \cdot w_3}{v_1 \cdot v_1} \cdot v_1 - \frac{v_2 \cdot w_3}{v_2 \cdot v_2} \cdot v_2$

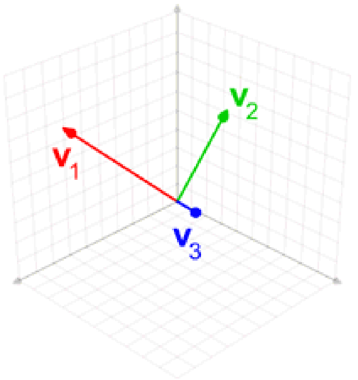


- Projection on previous vectors
- Subtract the projected part

Exercise 7.2: Gram-Schmidt Orthogonalization

- Orthogonalization
  - Given: Linearly independent vectors $w_1, \dots, w_3$
  - Unknown: Pairwise orthogonal vectors $v_1, \dots, v_3$ that span the same subspace

- Gram-Schmidt
  - $v_1 = w_1 = \begin{pmatrix} 0.6 \\ 0.5 \\ 0.1 \end{pmatrix}$
  - $v_2 = w_2 - \frac{v_1 \cdot w_2}{v_1 \cdot v_1} \cdot v_1 \approx \begin{pmatrix} -0.394 \\ 0.489 \\ -0.082 \end{pmatrix}$
  - $v_3 = w_3 - \frac{v_1 \cdot w_3}{v_1 \cdot v_1} \cdot v_1 - \frac{v_2 \cdot w_3}{v_2 \cdot v_2} \cdot v_2 \approx \begin{pmatrix} -0.122 \\ 0.013 \\ 0.669 \end{pmatrix}$



- Projection on previous vectors
- Subtract the projected part

Exercise 7.2: Normalization

Normalization:

$$e_1 = \frac{v_1}{\|v_1\|} \approx \frac{1}{\sqrt{0.62}} \cdot \begin{pmatrix} 0.6 \\ 0.5 \\ 0.1 \end{pmatrix} \approx \begin{pmatrix} 0.762 \\ 0.635 \\ 0.127 \end{pmatrix}$$

$$e_2 = \frac{v_2}{\|v_2\|} \approx \frac{1}{\sqrt{0.401}} \cdot \begin{pmatrix} -0.394 \\ 0.489 \\ -0.082 \end{pmatrix} \approx \begin{pmatrix} -0.622 \\ 0.772 \\ -0.129 \end{pmatrix}$$

$$e_3 = \frac{v_3}{\|v_3\|} \approx \frac{1}{\sqrt{0.463}} \cdot \begin{pmatrix} -0.122 \\ 0.013 \\ 0.669 \end{pmatrix} \approx \begin{pmatrix} -0.179 \\ 0.019 \\ 0.984 \end{pmatrix}$$

Exercise 7.2: Result

- Input:

$$A = \begin{pmatrix} 0.6 & 0.1 & 0.1 \\ 0.5 & 0.9 & 0.5 \\ 0.1 & 0.0 & 0.7 \end{pmatrix}$$

- Orthonormal basis vectors:

$$e_1 = \begin{pmatrix} 0.762 \\ 0.635 \\ 0.127 \end{pmatrix}, e_2 = \begin{pmatrix} -0.622 \\ 0.772 \\ -0.129 \end{pmatrix}, e_3 = \begin{pmatrix} -0.179 \\ 0.019 \\ 0.984 \end{pmatrix}$$

- Rotation matrix:

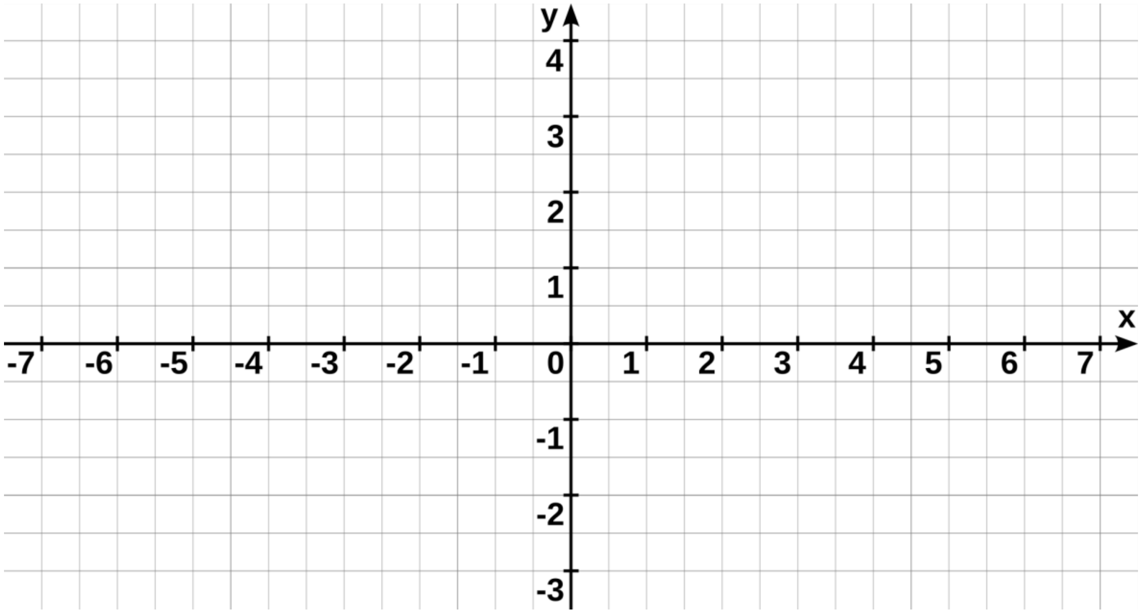
$$\mathrm{R} = \begin{pmatrix} 0.762 & -0.622 & -0.179 \\ 0.635 & 0.772 & 0.019 \\ 0.127 & -0.129 & 0.984 \end{pmatrix}$$

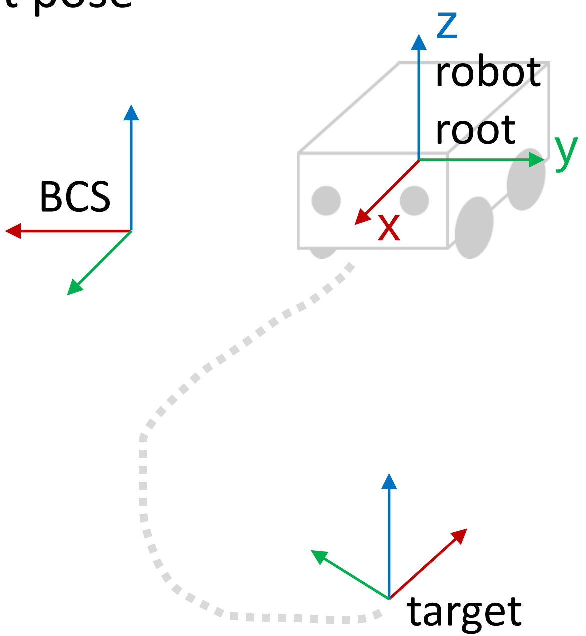
Python for Next Exercises

- Install python (≥ 3.6) and an IDE
  - E.g., PyCharm Community Edition (Linux/Windows/macOS) https://www.jetbrains.com/pycharm/download#community-edition
- Robotics Toolbox (Peter Corke)
  - https://github.com/petercorke/robotics-toolbox-python
  - Install via git repository or, recommended, via PyPI:
    ```
    pip3 install roboticstoolbox-python[collision]
    ```
- Give it a try: Solve exercise sheet 1 in python

Exercise 1: Transformations

- Robot with root pose, basis coordinate system, target pose





Exercise 1.1: Interpretation of a Pose

- Let the initial root pose of a robot, described in the *BCS*, be given as

$$^{BCS}T_{root} = \begin{pmatrix} \frac{1}{2}\sqrt{3} & -\frac{1}{2} & 0 & 3 \\ \frac{1}{2} & \frac{1}{2}\sqrt{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

1. Which transformation does $^{BCS}T_{root}$ describe?
2. At which position is the robot's root located?
   Draw the position in the *BCS*.

Exercise 1.1 (i): Interpretation of a Pose

$$^{BCS}T_{root} = \begin{pmatrix} \frac{1}{2}\sqrt{3} & -\frac{1}{2} & 0 & 3 \\ \frac{1}{2} & \frac{1}{2}\sqrt{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

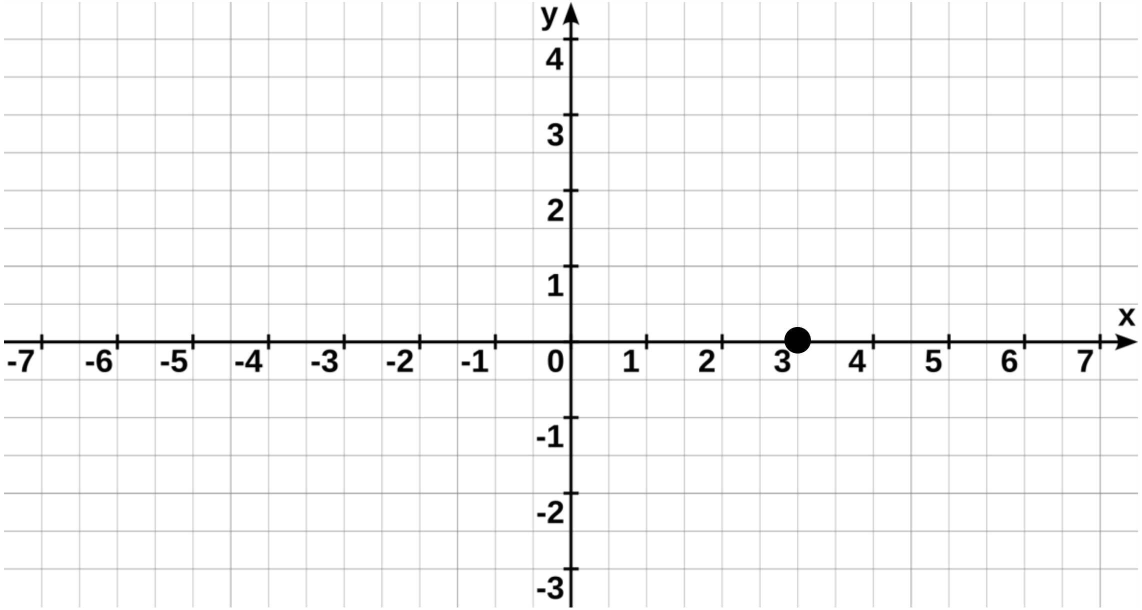
Transformation described by $^{BCS}T_{root}$:

- Translation by 3 along x-axis
- Rotation around z-axis, by 30°

α	0°	30°	60°	90°
$\sin(\alpha)$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos(\alpha)$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0

Exercise 1.1 (ii): Interpretation of a Pose

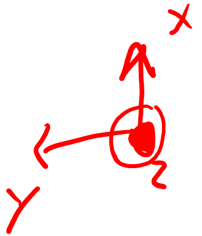
- Robot position: $(3, 0, 0)^\top$



Exercise 1.1 (iv): Interpretation of a Pose

- Vectors (in the robot's root coordinate system):

$$v_{x,root} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_{y,root} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad v_{z,root} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad v_{o,root} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$



Exercise 1.1 (iv): Interpretation of a Pose

- Vectors (in the robot's root coordinate system):

$$v_{x,root} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_{y,root} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad v_{z,root} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad v_{o,root} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

- Vectors (in the BCS):

$$v_{BCS} = {}^{BCS}T_{root} \cdot v_{root}$$

Exercise 1.1 (iv): Interpretation of a Pose

- Vectors (in the robot's root coordinate system):

$$v_{x,root} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_{y,root} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad v_{z,root} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad v_{o,root} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

- Vectors (in the BCS):

$$v_{BCS} = {}^{BCS}T_{root} \cdot v_{root}$$

$$\begin{pmatrix} \frac{1}{2}\sqrt{3} & -\frac{1}{2} & 0 & 3 \\ \frac{1}{2} & \frac{1}{2}\sqrt{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 + \frac{1}{2}\sqrt{3} \\ \frac{1}{2} \\ 0 \\ 1 \end{pmatrix} \Longrightarrow v_{x,BCS} = \begin{pmatrix} 3 + \frac{1}{2}\sqrt{3} \\ \frac{1}{2} \\ 0 \end{pmatrix} \quad \text{etc.}$$

Exercise 1.1 (iv): Interpretation of a Pose

- Vectors (in the robot's root coordinate system):

$$v_{x,root} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_{y,root} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad v_{z,root} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad v_{o,root} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

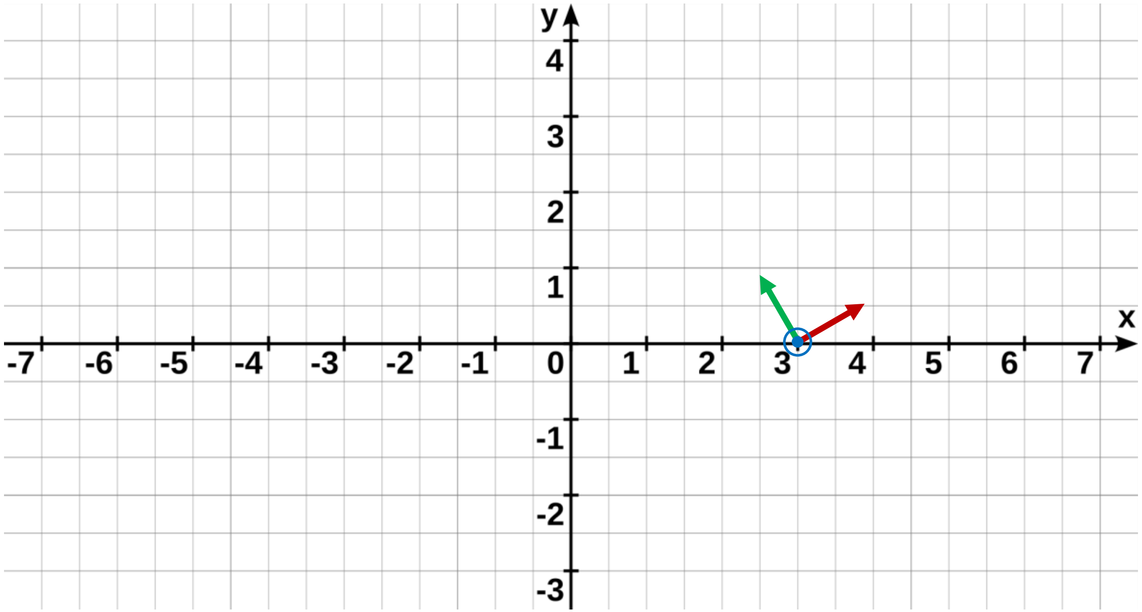
- Vectors (in the BCS):

$$v_{BCS} = {}^{BCS}T_{root} \cdot v_{root}$$

$$v_{x,BCS} = \begin{pmatrix} 3 + \frac{1}{2}\sqrt{3} \\ \frac{1}{2} \\ 0 \end{pmatrix}, \quad v_{y,BCS} = \begin{pmatrix} \frac{5}{2} \\ \frac{1}{2}\sqrt{3} \\ 0 \end{pmatrix}, \quad v_{z,BCS} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}, \quad v_{o,BCS} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$$

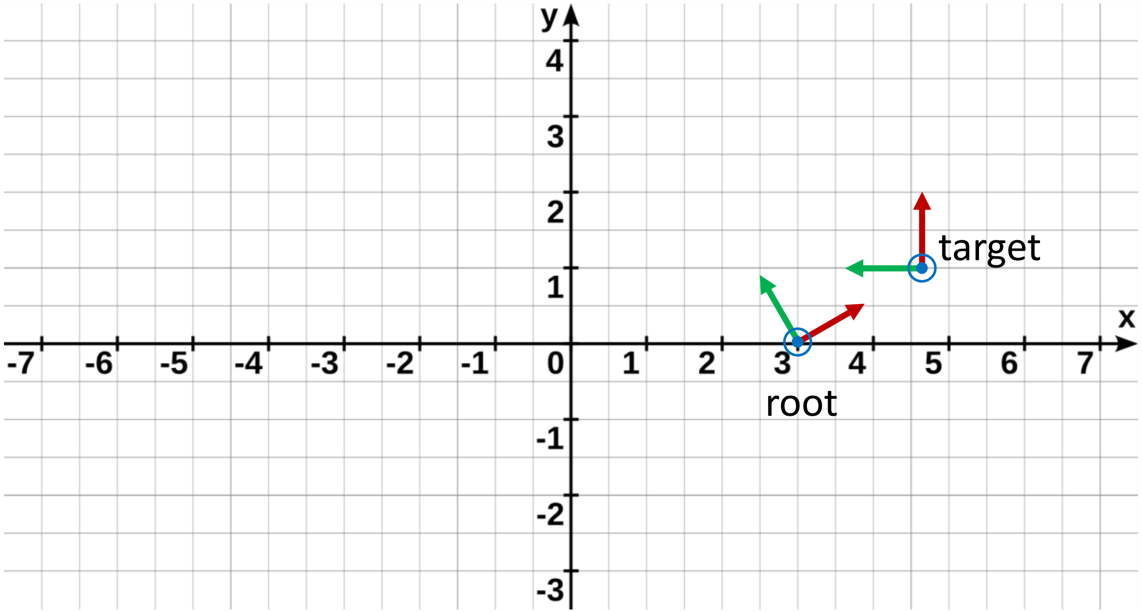
Exercise 1.1 (iv): Interpretation of a Pose

- Robot root pose ${}^{BCS}T_{root}$



Exercise 1.2 (i): Conversion Between Coordinate Systems

- Target pose ${}^{BCS}T_{target}$



$${}^{BCS}T_{target} =$$

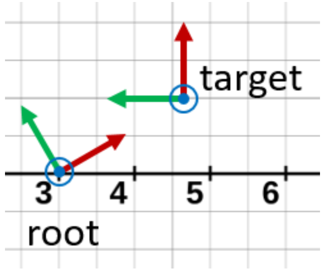
$$\begin{pmatrix} 0 & -1 & 0 & 3+\sqrt{3} \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

x-axis of target:
in y-direction of BCS

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

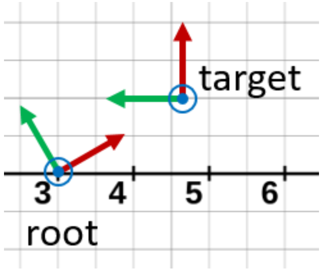
Exercise 1.2 (ii): Conversion Between Coordinate Systems

- Target pose in the robot's root coordinate system
  - Searched: ${}^{root}T_{target}$
  - Known relation: ${}^{BCS}T_{root} \cdot {}^{root}T_{target} = {}^{BCS}T_{target}$



Exercise 1.2 (ii): Conversion Between Coordinate Systems

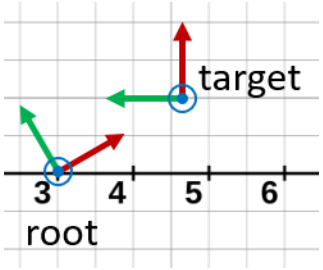
- Target pose in the robot's root coordinate system
  - Searched: ${}^{root}T_{target}$
  - Known relation: ${}^{BCS}T_{root} \cdot {}^{root}T_{target} = {}^{BCS}T_{target}$



$$\Leftrightarrow {}^{root}T_{target} = \left({}^{BCS}T_{root}\right)^{-1} \cdot {}^{BCS}T_{target}$$

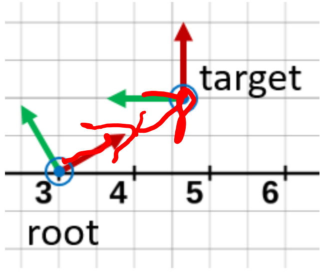
Exercise 1.2 (ii): Conversion Between Coordinate Systems

Reminder: $\begin{pmatrix} R & \boldsymbol{t} \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} R^\top & -R^\top \boldsymbol{t} \\ 0 & 1 \end{pmatrix}$



$$\left({}^{BCS}T_{root}\right)^{-1} = \begin{pmatrix} \frac{1}{2}\sqrt{3} & -\frac{1}{2} & 0 & 3 \\ \frac{1}{2} & \frac{1}{2}\sqrt{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{2}\sqrt{3} & \frac{1}{2} & 0 & -\frac{3}{2}\sqrt{3} \\ -\frac{1}{2} & \frac{1}{2}\sqrt{3} & 0 & \frac{3}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Exercise 1.2 (ii): Conversion Between Coordinate Systems

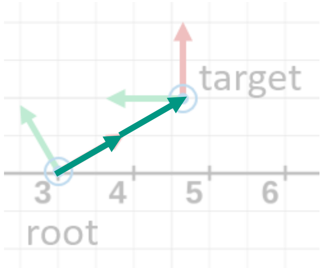


$$^{root}T_{target} = \left(^{BCS}T_{root}\right)^{-1} \cdot {}^{BCS}T_{target}$$

$$= \begin{pmatrix} \frac{1}{2}\sqrt{3} & \frac{1}{2} & 0 & -\frac{3}{2}\sqrt{3} \\ -\frac{1}{2} & \frac{1}{2}\sqrt{3} & 0 & \frac{3}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 & 3+\sqrt{3} \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{3} & 0 & \frac{3}{2}\sqrt{3} + \frac{3}{2} + \frac{1}{2} - \frac{3}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} & 0 & -\frac{3}{2} - \frac{1}{2}\sqrt{3} + \frac{1}{2}\sqrt{3} + \frac{3}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{3} & 0 & 2 \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Exercise 1.2 (ii): Conversion Between Coordinate Systems



$$ {}^{root}T_{target} = \left({}^{BCS}T_{root}\right)^{-1} \cdot {}^{BCS}T_{target} $$

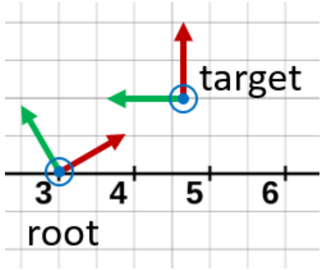
$$ = \begin{pmatrix} \frac{1}{2}\sqrt{3} & \frac{1}{2} & 0 & -\frac{3}{2}\sqrt{3} \\ -\frac{1}{2} & \frac{1}{2}\sqrt{3} & 0 & \frac{3}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 & 3+\sqrt{3} \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

$$ = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{3} & 0 & \frac{3}{2}\sqrt{3}+\frac{3}{2}+\frac{1}{2}-\frac{3}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} & 0 & -\frac{3}{2}-\frac{1}{2}\sqrt{3}+\frac{1}{2}\sqrt{3}+\frac{3}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{3} & 0 & 2 \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

Exercise 1.2 (iii): Conversion Between Coordinate Systems

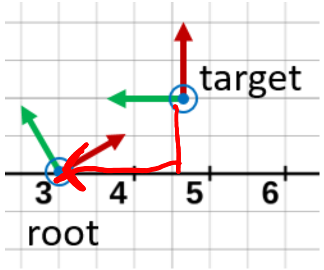
- Robot root pose in the target's coordinate system
  - Searched: ${}^{target}T_{root}$
  - Known relation: ${}^{BCS}T_{target} \cdot {}^{target}T_{root} = {}^{BCS}T_{root}$

$$\Leftrightarrow {}^{target}T_{root} = \left({}^{BCS}T_{target}\right)^{-1} \cdot {}^{BCS}T_{root}$$



Exercise 1.2 (iii): Conversion Between Coordinate Systems

- Robot root pose in the target's coordinate system
  - Searched: ${}^{target}T_{root}$
  - Known relation: ${}^{BCS}T_{target} \cdot {}^{target}T_{root} = {}^{BCS}T_{root}$



$$\Leftrightarrow {}^{target}T_{root} = \left({}^{BCS}T_{target}\right)^{-1} \cdot {}^{BCS}T_{root}$$

$$= \ldots = \begin{pmatrix} \frac{1}{2} & \frac{1}{2}\sqrt{3} & 0 & -1 \\ -\frac{1}{2}\sqrt{3} & \frac{1}{2} & 0 & \sqrt{3} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Small Exercise

- To calculate ${}^{root}T_{target}$, we used ${}^{BCS}T_{root} \cdot {}^{root}T_{target} = {}^{BCS}T_{target}$ as a starting point.
  Could we alternatively have used ${}^{BCS}T_{target} \cdot {}^{target}T_{root} = {}^{BCS}T_{root}$?

a) Yes, ${}^{target}T_{root}$ and ${}^{root}T_{target}$ are the same.

b) Yes, together with ${}^{root}T_{target} = \left({}^{target}T_{root}\right)^{-1}$.

c) Yes, together with ${}^{root}T_{target} = \left({}^{target}T_{root}\right)^{\mathsf{T}}$.

d) No, these equations describe completely different situations.

Small Exercise

- To calculate ${}^{root}T_{target}$, we used ${}^{BCS}T_{root} \cdot {}^{root}T_{target} = {}^{BCS}T_{target}$ as a starting point.
  Could we alternatively have used ${}^{BCS}T_{target} \cdot {}^{target}T_{root} = {}^{BCS}T_{root}$?

a) Yes, ${}^{target}T_{root}$ and ${}^{root}T_{target}$ are the same.

b) **Yes, together with $\boldsymbol{{}^{root}T_{target} = \left({}^{target}T_{root}\right)^{-1}}$.**

c) Yes, together with ${}^{root}T_{target} = \left({}^{target}T_{root}\right)^{\top}$.

d) No, these equations describe completely different situations.

Small Exercise

$$\left({}^{target}T_{root}\right)^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2}\sqrt{3} & 0 & -1 \\ -\frac{1}{2}\sqrt{3} & \frac{1}{2} & 0 & \sqrt{3} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{3} & 0 & \frac{1}{2}+\frac{3}{2} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} & 0 & \frac{1}{2}\sqrt{3}-\frac{1}{2}\sqrt{3} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{3} & 0 & 2 \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = {}^{root}T_{target} \text{ as previously calculated}$$

Exercise 1.3 (i): Local and Global Transformation

- Transformation T corresponding to 60° rotation around z-axis:

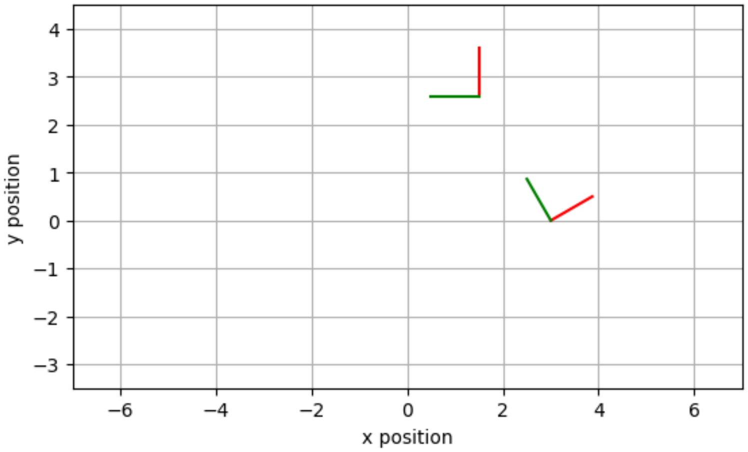
$$T = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{3} & 0 & 0 \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

α	$0°$	$30°$	$60°$	$90°$
$\sin(\alpha)$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos(\alpha)$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0

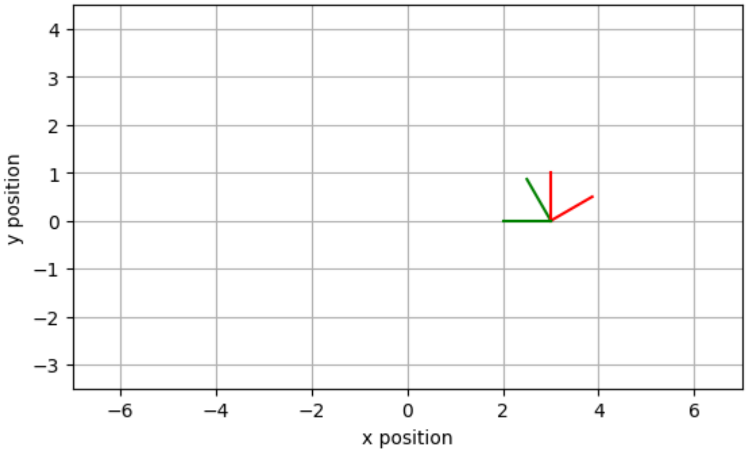
Exercise 1.3 (ii-iv): Local and Global Transformation

- Application from the left and from the right:

$$T \cdot {}^{BCS}T_{root}$$



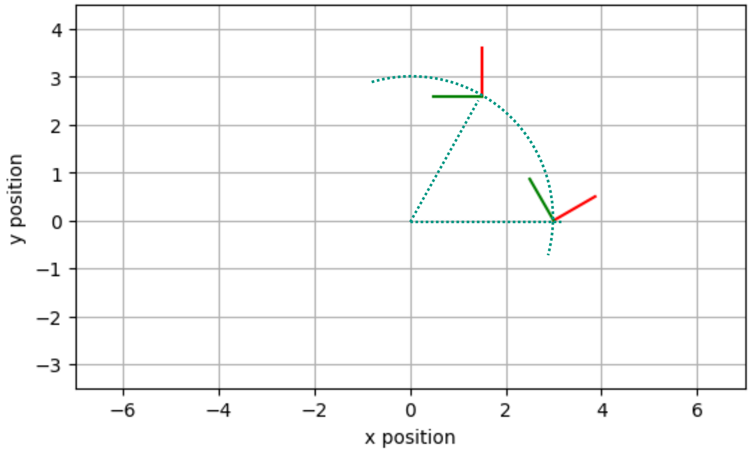
$${}^{BCS}T_{root} \cdot T$$



Exercise 1.3 (ii-iv): Local and Global Transformation

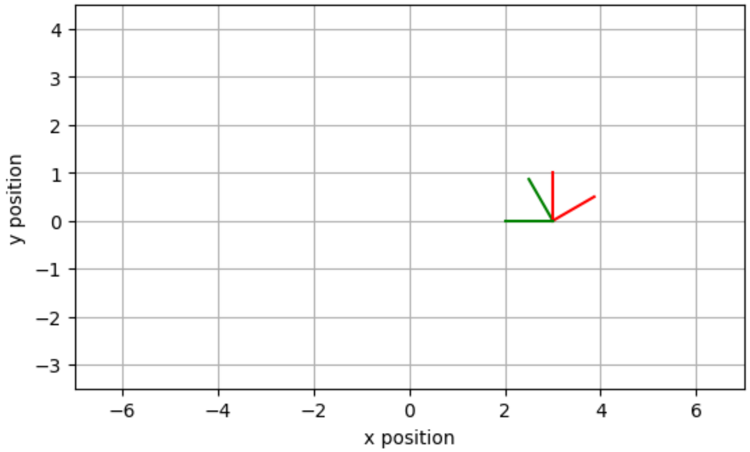
- Application from the left and from the right:

$$T \cdot {}^{BCS}T_{root}$$



Global transformation

$$^{BCS}T_{root} \cdot T$$

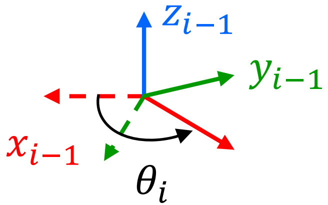


Local transformation

DH Transformation Matrices (1)

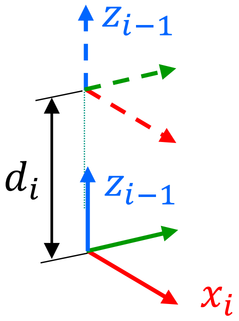
Transformation from LCS_{i-1} to LCS_i

1. A **rotation θ_i around the z_{i-1}-axis** so that the x_{i-1}-axis is parallel to the x_i-axis.



$$R_{z_{i-1}}(\theta_i) = \begin{pmatrix} \cos\theta_i & -\sin\theta_i & 0 & 0 \\ \sin\theta_i & \cos\theta_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

2. A **translation d_i** along the **z_{i-1}-axis** to the point where z_{i-1} and x_i intersect.



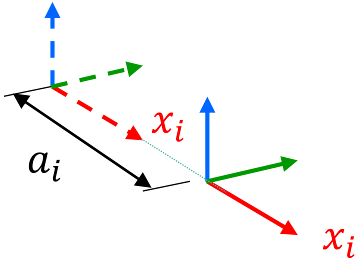
$$T_{z_{i-1}}(d_i) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

LCS_i: *Local Coordinate System of joint i*

DH Transformation Matrices (2)

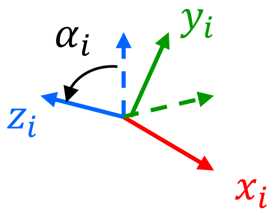
Transformation from LCS_{i-1} to LCS_i

3. A **translation** a_i along the x_i**-axis** to align the origins of the coordinate systems.



$$T_{x_i}(a_i) = \begin{pmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

4. A **rotation** α_i around the x_i**-axis** to convert the z_{i-1}-axis into the z_i-axis.



$$R_{x_i}(\alpha_i) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha_i & -\sin\alpha_i & 0 \\ 0 & \sin\alpha_i & \cos\alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

LCS_i: *Local Coordinate System of joint i*

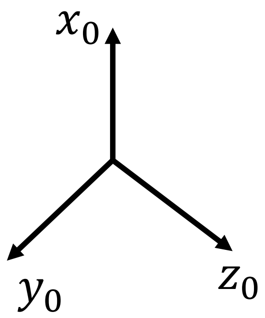
DH Transformation Matrices (3)

Transformation from LCS_{i-1} to LCS_i

$$A_{i-1,i} = R_{z_{i-1}}(\theta_i) \cdot T_{z_{i-1}}(d_i) \cdot T_{x_i}(a_i) \cdot R_{x_i}(\alpha_i)$$
$$= \begin{bmatrix} \cos\theta_i & -\sin\theta_i \cdot \cos\alpha_i & \sin\theta_i \cdot \sin\alpha_i & a_i \cdot \cos\theta_i \\ \sin\theta_i & \cos\theta_i \cdot \cos\alpha_i & -\cos\theta_i \cdot \sin\alpha_i & a_i \cdot \sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

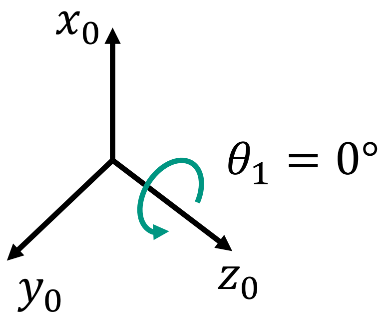
Exercise 2.1 (i): DH Transformation

- DH parameters: $\theta_1 = 0°, d_1 = 60\text{ mm}, a_1 = 0\text{ mm}, \alpha_1 = 180°$
- Rotation around z_0-axis by $\theta_1 = 0°$:



Exercise 2.1 (i): DH Transformation

- DH parameters: $\boldsymbol{\theta_1 = 0°}, d_1 = 60\ \text{mm}, a_1 = 0\ \text{mm}, \alpha_1 = 180°$
- Rotation around z_0-axis by $\theta_1 = 0°$:



Exercise 2.1 (i): DH Transformation

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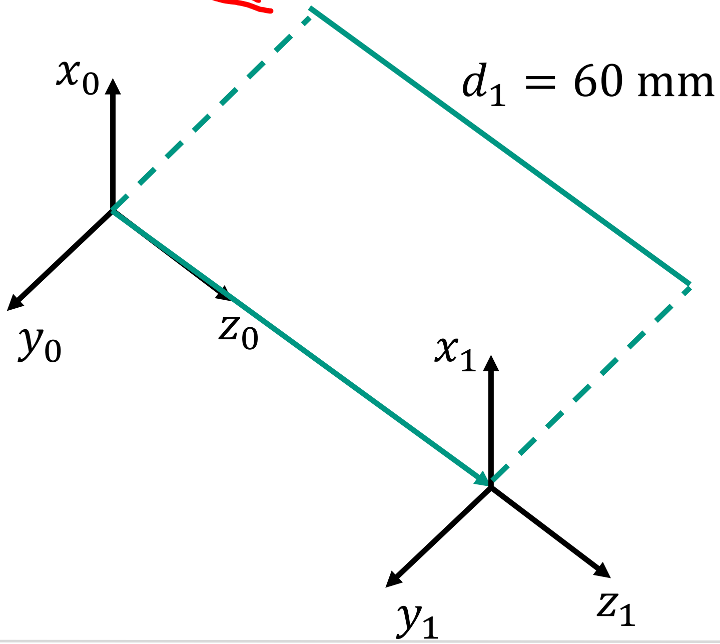
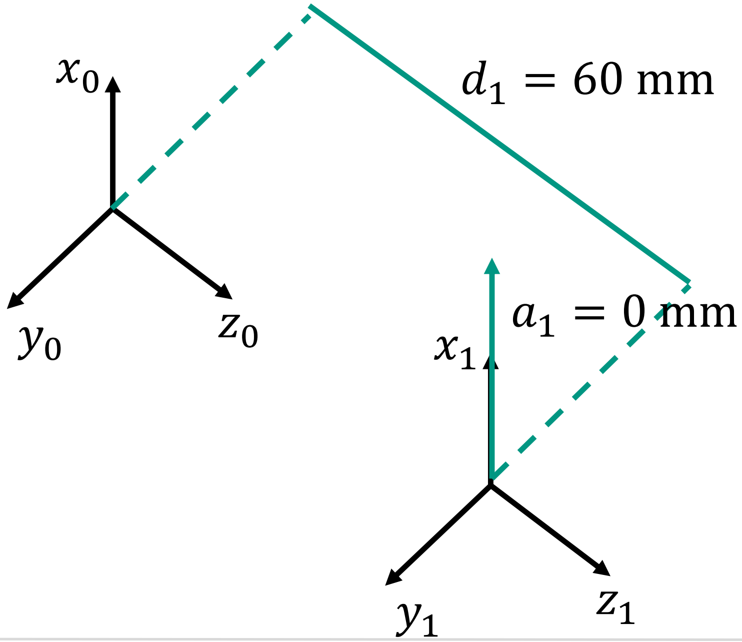
- DH parameters: $\theta_1 = 0°$, $\boldsymbol{d_1 = 60\ \mathrm{mm}}$, $a_1 = 0\ \mathrm{mm}$, $\alpha_1 = 180°$
- Translation along z_0-axis by $d_1 = 60\ \mathrm{mm}$:



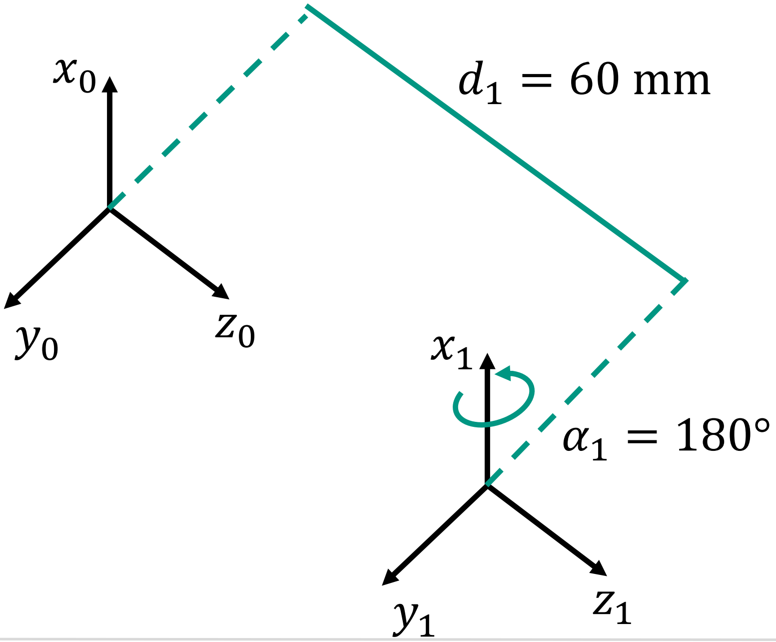
Exercise 2.1 (i): DH Transformation

- DH parameters: $\theta_1 = 0°, d_1 = 60\ mm,$ **$a_1 = 0\ mm$**, $\alpha_1 = 180°$
- Translation along x_1-axis by $a_1 = 0$ mm:



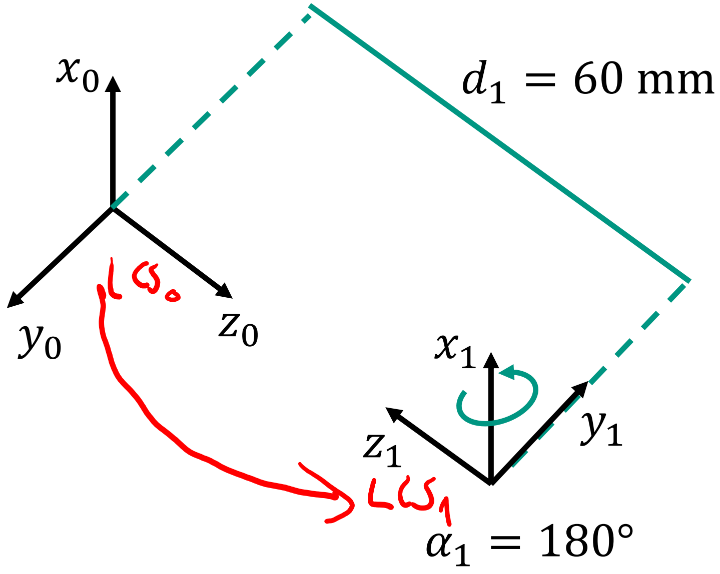
Exercise 2.1 (i): DH Transformation

- DH parameters: $\theta_1 = 0°, d_1 = 60\ mm, a_1 = 0\ mm,$ $\boldsymbol{\alpha_1 = 180°}$
- Rotation around x_1-axis by $\alpha_1 = 180°$:



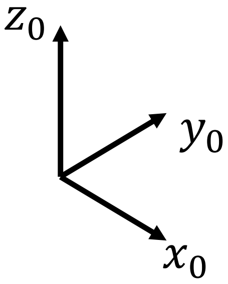
Exercise 2.1 (i): DH Transformation

- DH parameters: $\theta_1 = 0°, d_1 = 60\ mm, a_1 = 0\ mm, \boldsymbol{\alpha_1 = 180°}$
- Rotation around x_1-axis by $\alpha_1 = 180°$:



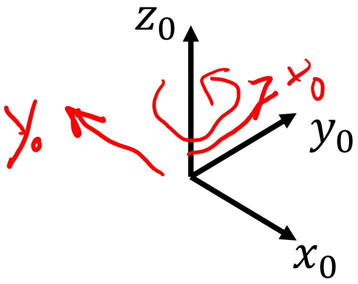
Exercise 2.1 (ii): DH Transformation

- DH parameters: $\theta_1 = 90°, d_1 = -30 \text{ mm}, a_1 = 60 \text{ mm}, \alpha_1 = -90°$



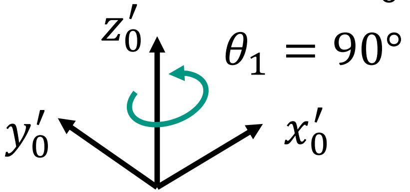
Exercise 2.1 (ii): DH Transformation

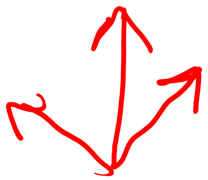
- DH parameters: $\boldsymbol{\theta_1 = 90°}$, $d_1 = -30$ mm, $a_1 = 60$ mm, $\alpha_1 = -90°$
- Rotation around z_0-axis by $\theta_1 = 90°$:



Exercise 2.1 (ii): DH Transformation

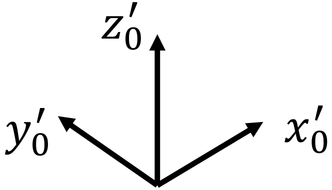
- DH parameters: $\boldsymbol{\theta_1 = 90°}$, $d_1 = -30\ \text{mm}, a_1 = 60\ \text{mm}, \alpha_1 = -90°$
- Rotation around z_0-axis by $\theta_1 = 90°$:





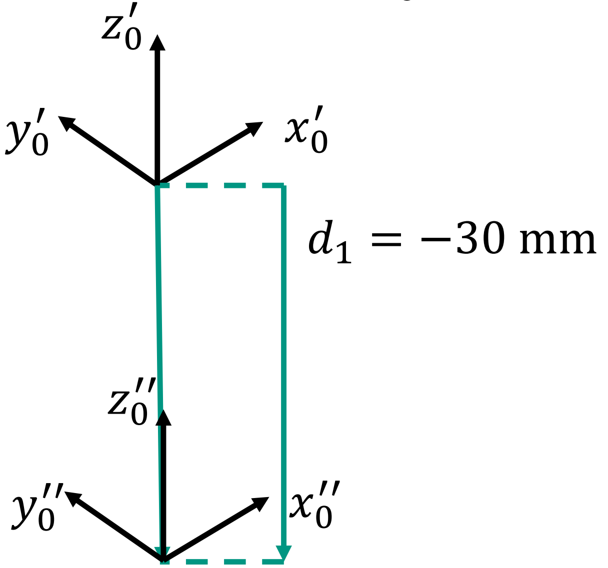
Exercise 2.1 (ii): DH Transformation

- DH parameters: $\theta_1 = 90°, \boldsymbol{d_1 = -30\ \mathrm{mm}}, a_1 = 60\ \mathrm{mm}, \alpha_1 = -90°$
- Translation along z_0-axis by $d_1 = -30\ mm$:



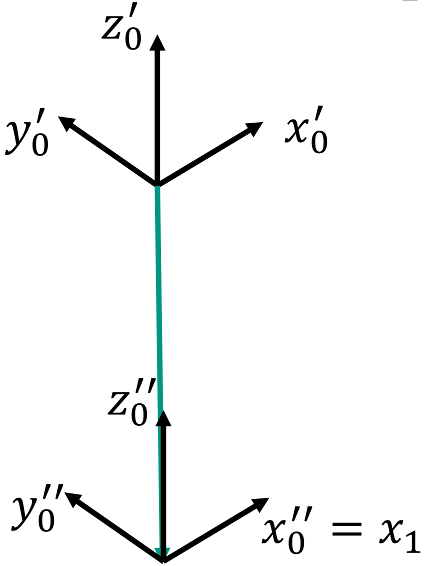
Exercise 2.1 (ii): DH Transformation

- DH parameters: $\theta_1 = 90°$, $\boldsymbol{d_1 = -30\ \mathrm{mm}}$, $a_1 = 60\ \mathrm{mm}$, $\alpha_1 = -90°$
- Translation along z_0-axis by $d_1 = -30\ mm$:



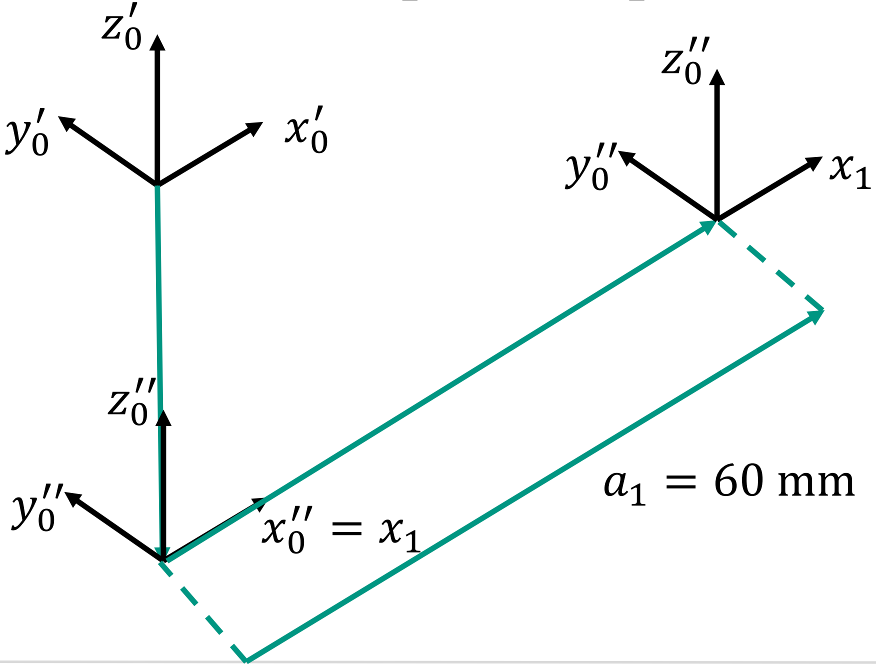
Exercise 2.1 (ii): DH Transformation

- DH parameters: $\theta_1 = 90°, d_1 = -30\text{ mm}, \boldsymbol{a_1 = 60\text{ mm}}, \alpha_1 = -90°$
- Translation along x_1-axis by $a_1 = 60\text{ mm}$:



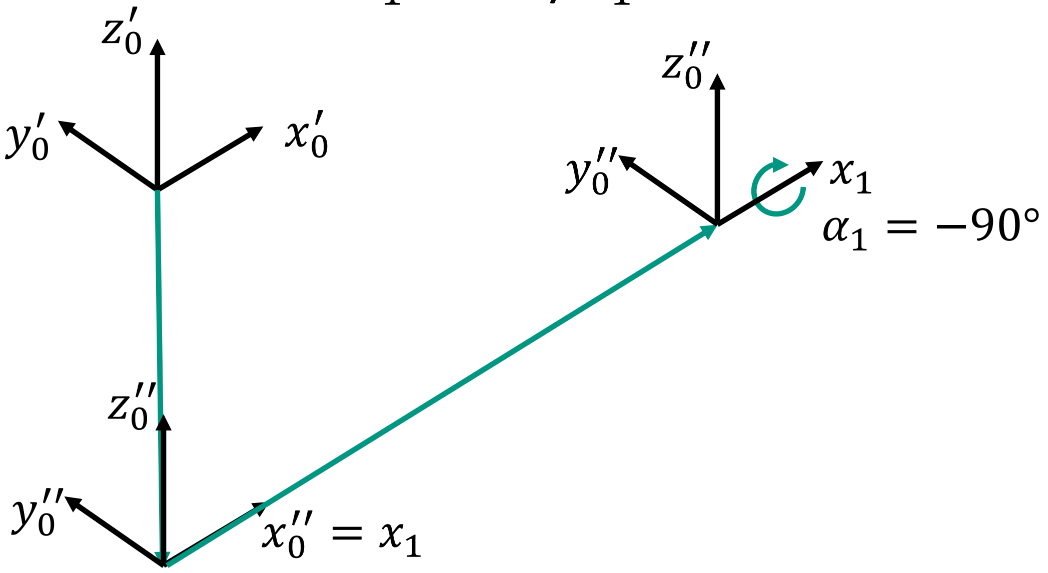
Exercise 2.1 (ii): DH Transformation

- DH parameters: $\theta_1 = 90°, d_1 = -30\text{ mm}, \boldsymbol{a_1 = 60\text{ mm}}, \alpha_1 = -90°$
- Translation along x_1-axis by $a_1 = 60\text{ mm}$:



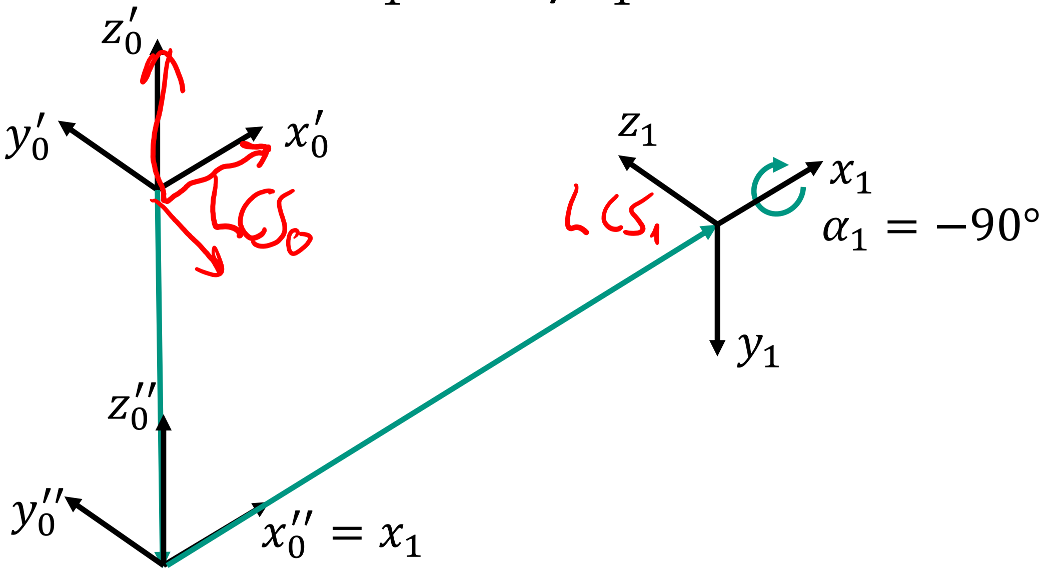
Exercise 2.1 (ii): DH Transformation

- DH parameters: $\theta_1 = 90°, d_1 = -30\text{ mm}, a_1 = 60\text{ mm}, \boldsymbol{\alpha_1 = -90°}$
- Rotation around x_1-axis by $\alpha_1 = -90°$:



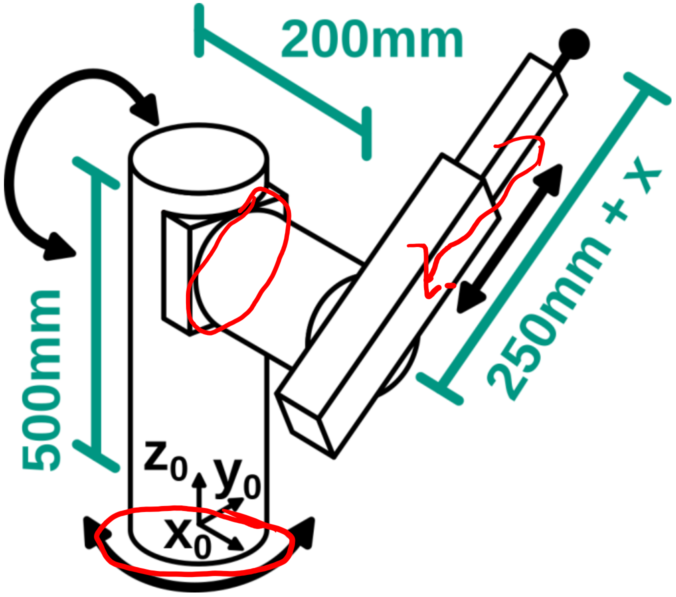
Exercise 2.1 (ii): DH Transformation

- DH parameters: $\theta_1 = 90°, d_1 = -30 \text{ mm}, a_1 = 60 \text{ mm}, \boldsymbol{\alpha_1 = -90°}$
- Rotation around x_1-axis by $\alpha_1 = -90°$:



Exercise 2.2: DH Parameters

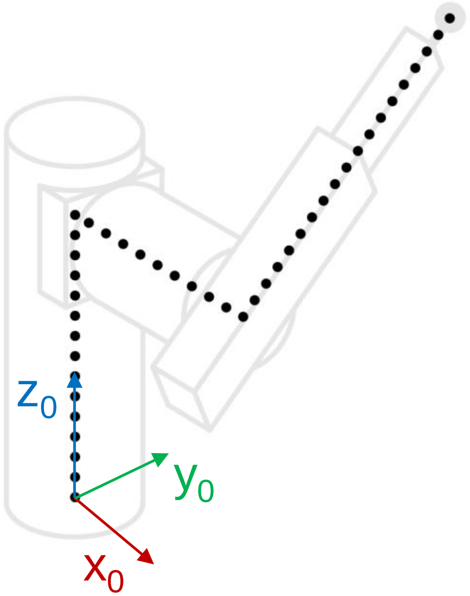
- Determine the DH parameters



Summary: Determination of the DH Parameters

1. **Sketch** of the manipulator
2. Identify and **enumerate** the **joints** (1, ..., last link = n)
3. Draw the **axes** z_{i-1} for **each joint *i***
4. Determine the **parameters** a_i between z_{i-1} and z_i
5. Draw the x_i**–axes**
6. Determine the **parameters** α_i (twist around the x_i-axes)
7. Determine the **parameters** d_i (link offset)
8. Determine the **angles** θ_i around the z_{i-1}-axes
9. Compose the **joint transformation matrices** $A_{i-1,i}$

Exercise 2.2: DH Parameters



1. **Sketch** of the manipulator
2. Identify and **enumerate** the **joints** (1, …, last link = n)
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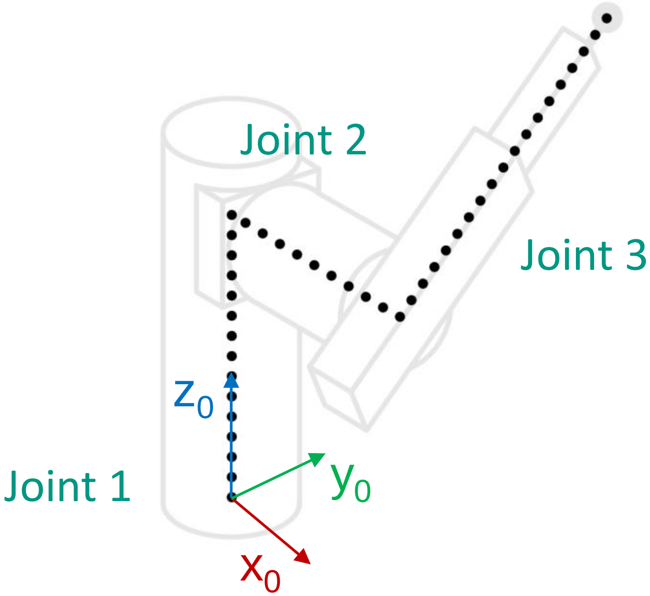
Exercise 2.2: DH Parameters



1. **Sketch** of the manipulator
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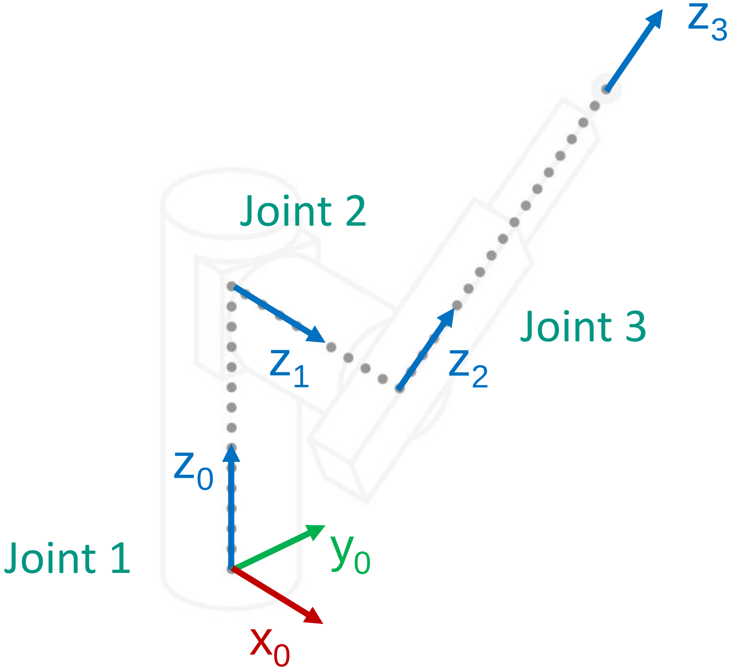
9. Compose the **joint transformation matrices** $A_{i-1,i}$

Exercise 2.2: DH Parameters

z_i runs along the **joint axis** $i + 1$



1. **Sketch** of the manipulator
2. Identify and **enumerate** the **joints** (1, …, last link = n)
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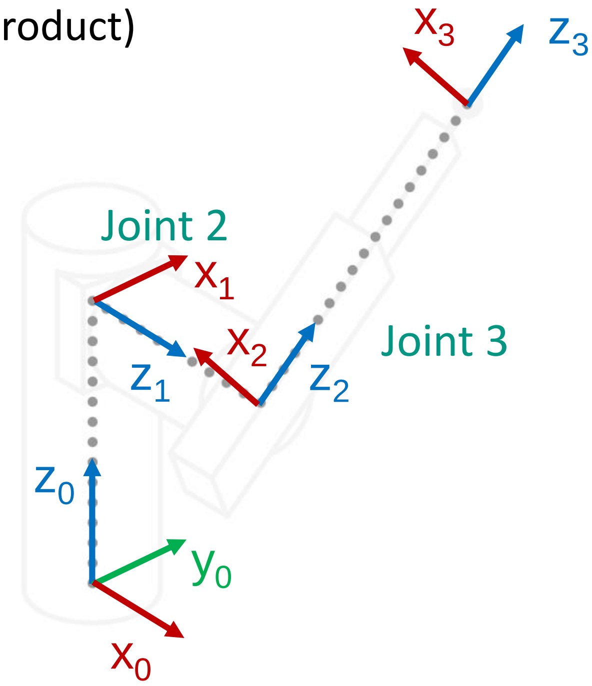
9. Compose the **joint transformation matrices** $A_{i-1,i}$

Exercise 2.2: DH Parameters

x_i lies along the **normal of z_{i-1} and z_i** (cross product)



1. **Sketch** of the manipulator
2. Identify and **enumerate** the **joints** (1, …, last link = n)
3. Draw the **axes** z_{i-1} for **each joint** i
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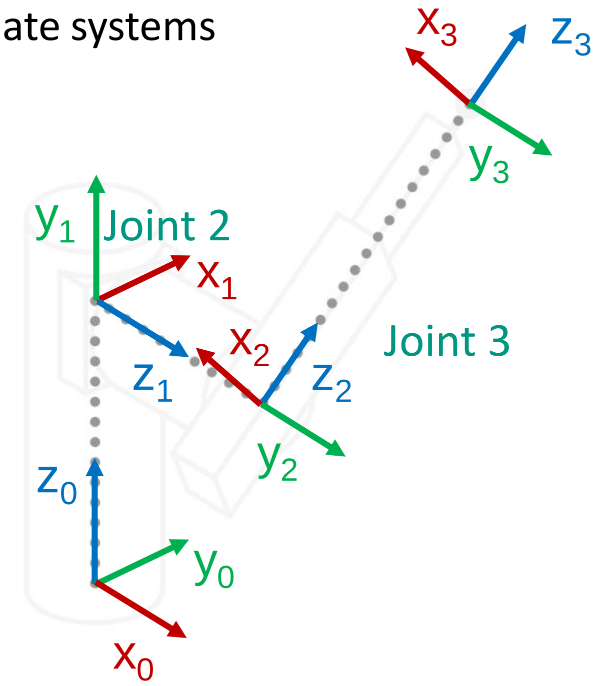
8. Determine the **angles** θ_i around the z_{i-1}-axes
9. Compose the **joint transformation matrices** $A_{i-1,i}$

Exercise 2.2: DH Parameters

y_i to get right-handed orthonormal coordinate systems



1. **Sketch** of the manipulator
2. Identify and **enumerate** the **joints** (1, …, last link = n)
3. Draw the **axes** z_{i-1} for **each joint** $\boldsymbol{i}$
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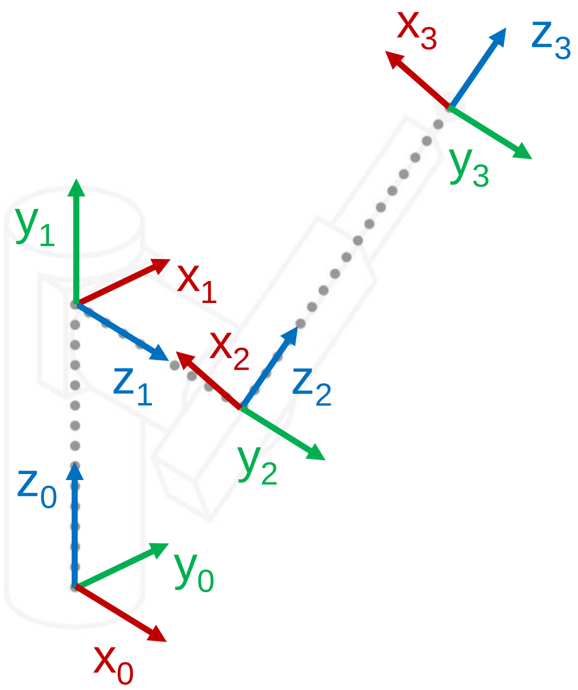
Exercise 2.2: DH Parameters



1. **Sketch** of the manipulator
2. Identify and **enumerate** the **joints** (1, ..., last link = n)
3. Draw the **axes** z_{i-1} for **each joint** i
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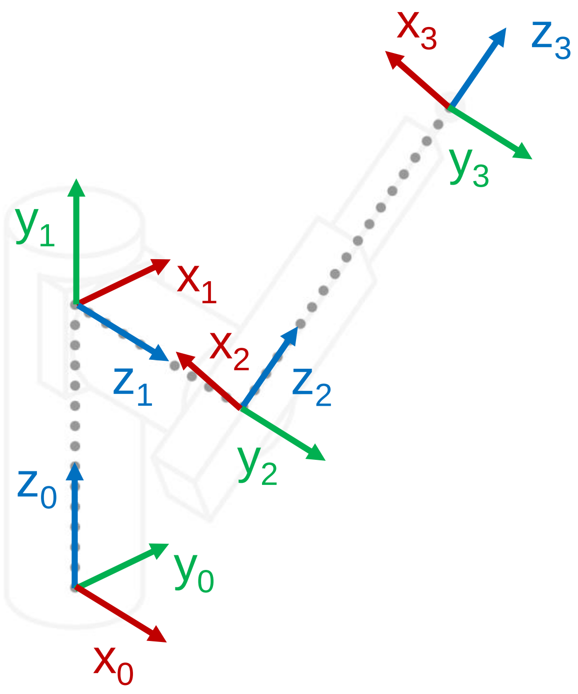
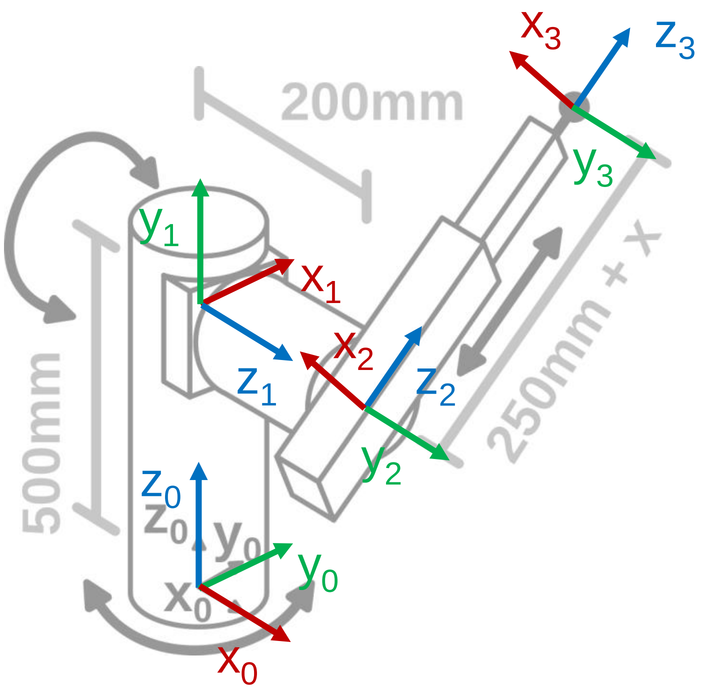
Exercise 2.2: DH Parameters



Joint angle θ_i is the angle from x_{i-1} to x_i **around z_{i-1}**

	θ	d	a	α
Joint 1	$90° + \theta_1$			
Joint 2	$90° + \theta_2$			
Joint 3	$0°$			

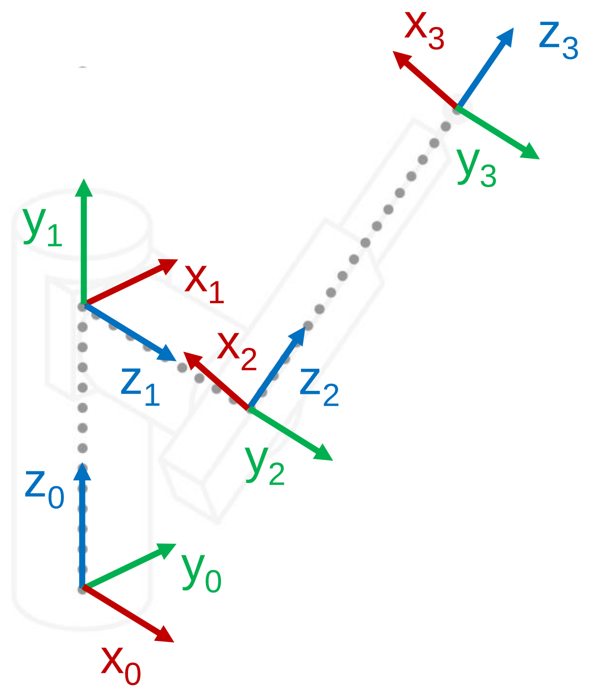
Exercise 2.2: DH Parameters



Link offset d_i is the distance between x_{i-1}-axis and x_i -axis **along the z_{i-1}**-axis

	θ	d	a	α
Joint 1	$90° + \theta_1$	$500\ \mathrm{mm}$		
Joint 2	$90° + \theta_2$	$200\ \mathrm{mm}$		
Joint 3	$0°$	$250\ \mathrm{mm} + \mathrm{x}$		

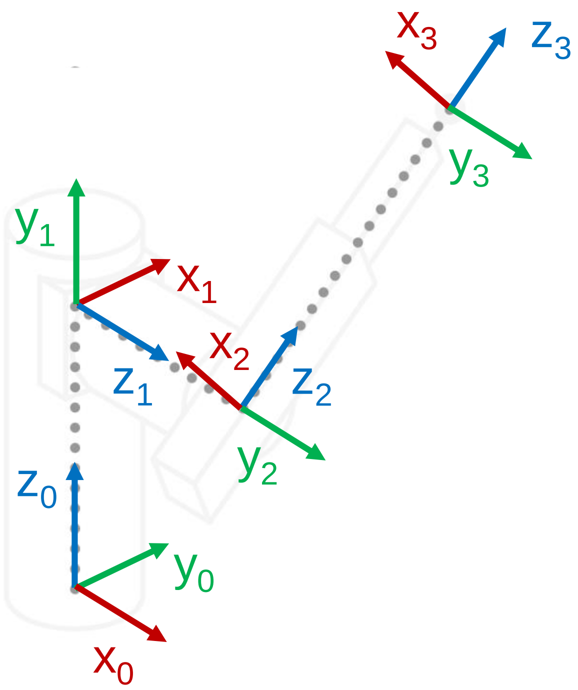
Exercise 2.2: DH Parameters



Link length a_i of an arm element i describes the **distance** from z_{i-1} to z_i along x_i

	θ	d	a	α
Joint 1	$90° + \theta_1$	$500\ \mathrm{mm}$	$0\ \mathrm{mm}$	
Joint 2	$90° + \theta_2$	$200\ \mathrm{mm}$	$0\ \mathrm{mm}$	
Joint 3	$0°$	$250\ \mathrm{mm} + \mathrm{x}$	$0\ \mathrm{mm}$	

Exercise 2.2: DH Parameters



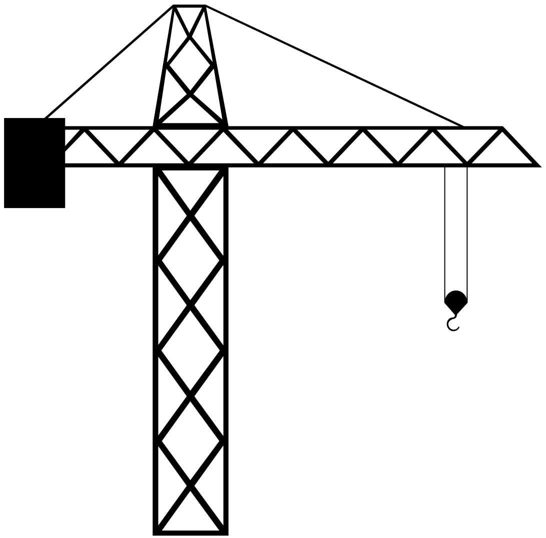
Link twist α_i describes the **angle** from z_{i-1} to z_i **around x_i**

	θ	d	a	α
Joint 1	$90° + \theta_1$	500 mm	0 mm	90°
Joint 2	$90° + \theta_2$	200 mm	0 mm	90°
Joint 3	0°	250 mm + x	0 mm	0°

Exercise 3: Crane

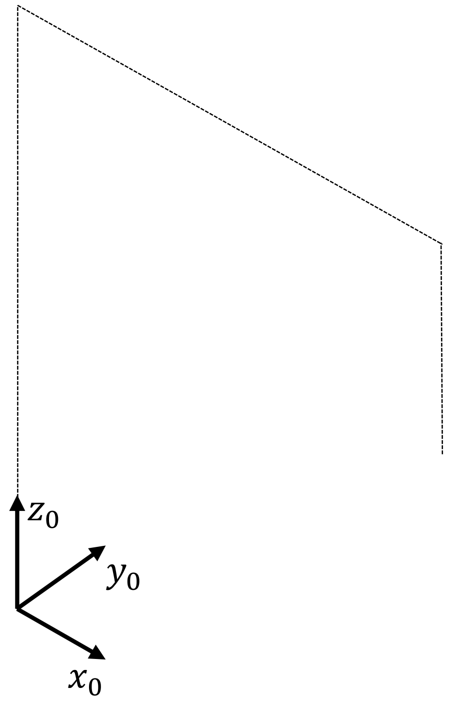
The crane can rotate by 360° and has a height of $20\ \mathrm{m}$ between ground and crane boom. The crane boom is $15\ \mathrm{m}$ long. The trolley stops $2\ \mathrm{m}$ away from the rotation axis. The hook can be lowered until reading the ground.

1. Determine the DH parameters of the crane and the resulting transformation matrix of the end effector.
2. Determine the Jacobian matrix of the end effector.
3. Determine the end effector velocities.

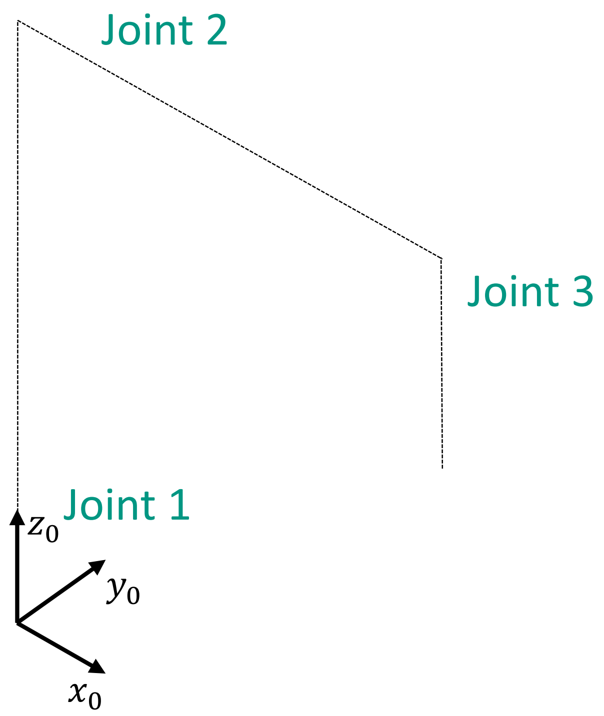


https://thenounproject.com/term/crane/2225/
mirrored (CC Attribution 3.0)

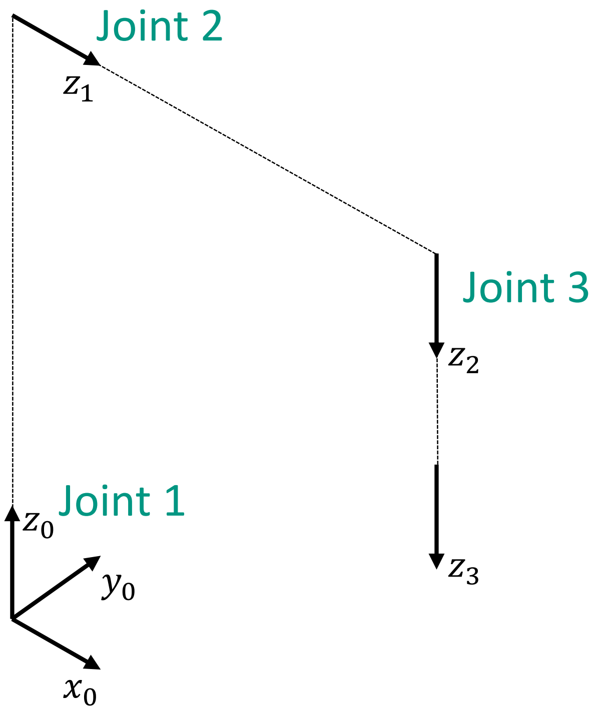
Exercise 3.1: DH Parameters of the Crane



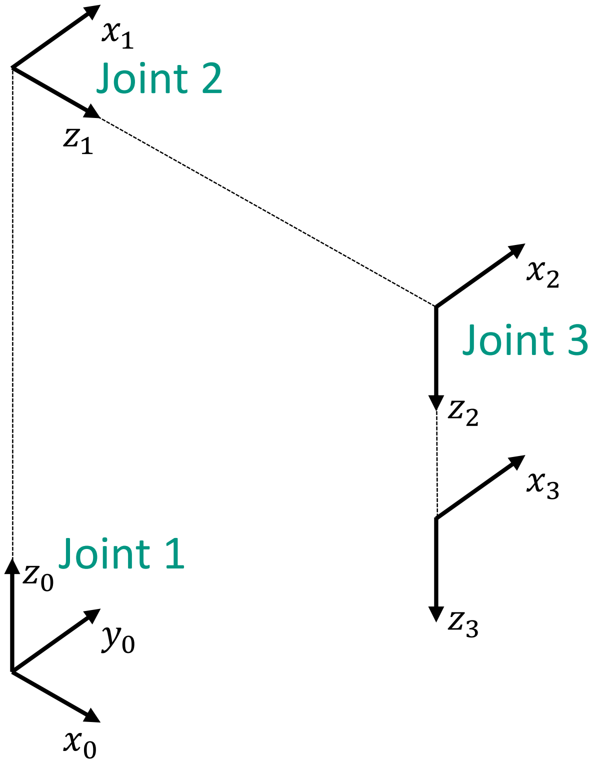
Exercise 3.1: DH Parameters of the Crane



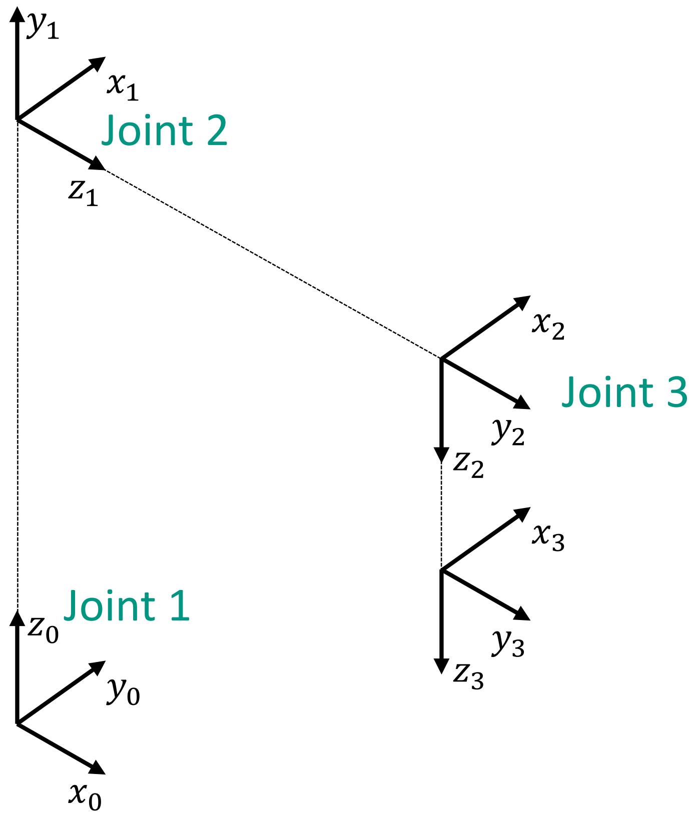
Exercise 3.1: DH Parameters of the Crane



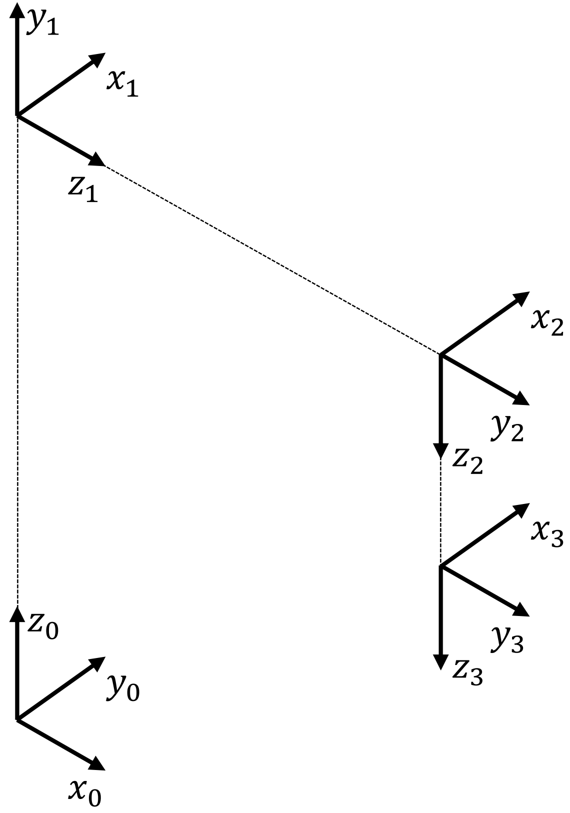
Exercise 3.1: DH Parameters of the Crane



Exercise 3.1: DH Parameters of the Crane

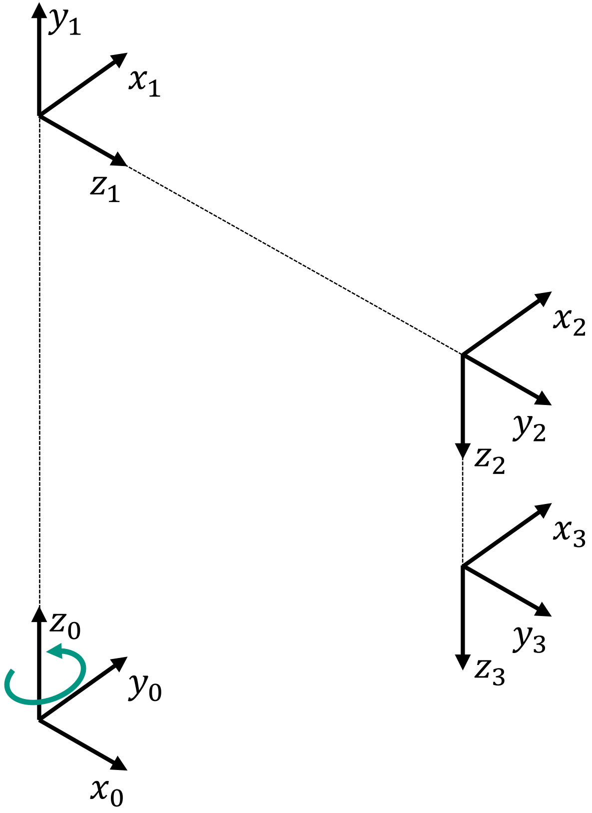


Exercise 3.1: DH Parameters of the Crane



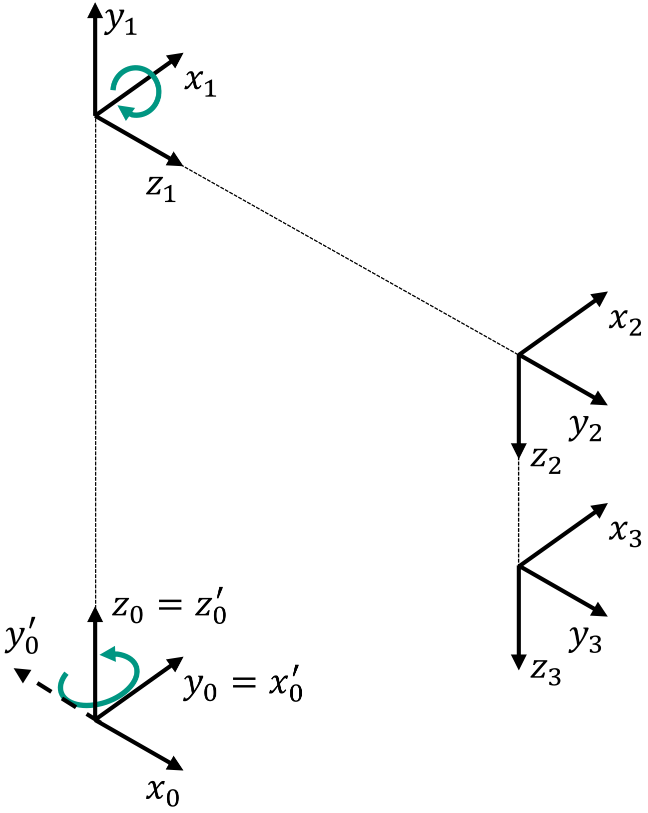
	θ	d	a	α
Joint 1				
Joint 2				
Joint 3				

Exercise 3.1: DH Parameters of the Crane



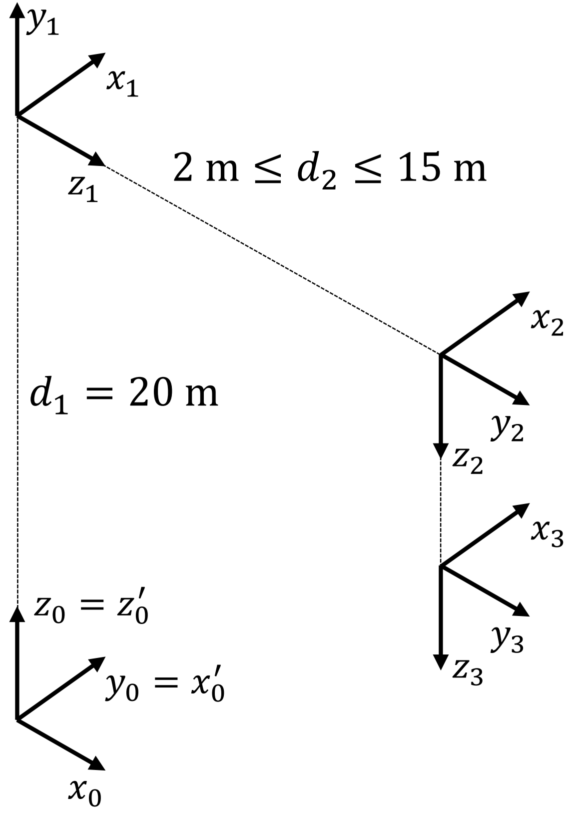
	θ	d	a	α
Joint 1	$\theta_1 + 90°$	20 m		
Joint 2				
Joint 3				

Exercise 3.1: DH Parameters of the Crane



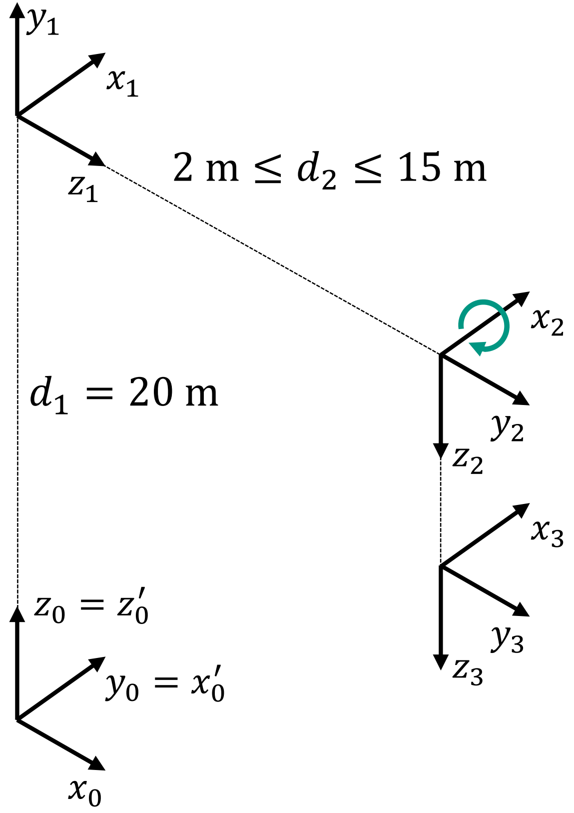
	θ	d	a	α
Joint 1	$\theta_1 + 90°$	20 m	0 m	90°
Joint 2				
Joint 3				

Exercise 3.1: DH Parameters of the Crane



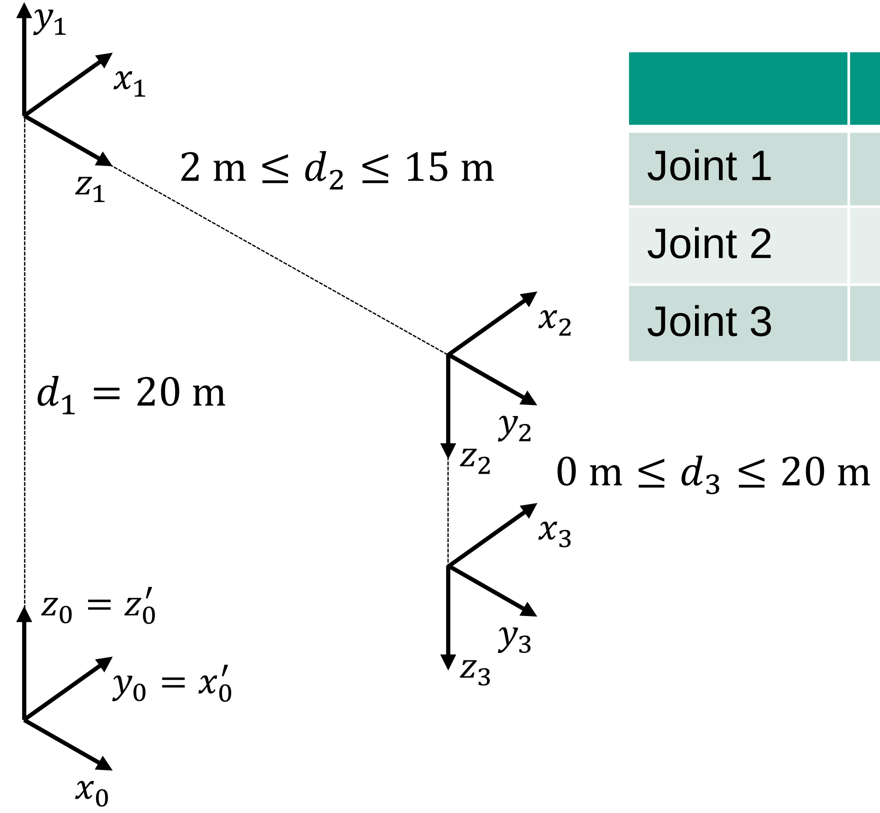
	θ	d	a	α
Joint 1	$\theta_1 + 90°$	20 m	0 m	$90°$
Joint 2	$0°$	$2\text{ m} \leq d_2 \leq 15\text{ m}$		
Joint 3				

Exercise 3.1: DH Parameters of the Crane



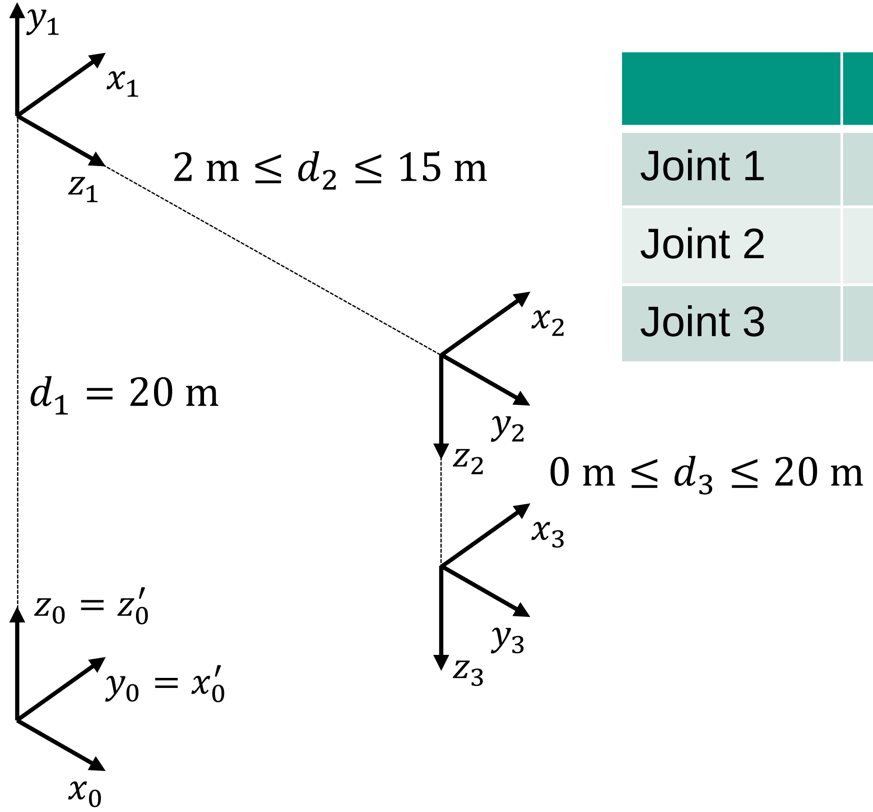
	θ	d	a	α
Joint 1	$\theta_1 + 90°$	20 m	0 m	$90°$
Joint 2	$0°$	$2\text{ m} \leq d_2 \leq 15\text{ m}$	0 m	$90°$
Joint 3				

Exercise 3.1: DH Parameters of the Crane



	θ	d	a	α
Joint 1	$\theta_1 + 90°$	$20\ \mathrm{m}$	$0\ \mathrm{m}$	$90°$
Joint 2	$0°$	$2\ \mathrm{m} \leq d_2 \leq 15\ \mathrm{m}$	$0\ \mathrm{m}$	$90°$
Joint 3	$0°$	$0\ \mathrm{m} \leq d_3 \leq 20\ \mathrm{m}$		

Exercise 3.1: DH Parameters of the Crane



	θ	d	a	α
Joint 1	$\theta_1 + 90°$	20 m	0 m	$90°$
Joint 2	$0°$	$2\text{ m} \leq d_2 \leq 15\text{ m}$	0 m	$90°$
Joint 3	$0°$	$0\text{ m} \leq d_3 \leq 20\text{ m}$	0 m	$0°$

Exercise 3.1: Transformation Matrix of the Crane

	θ	d	a	α
Joint 1	$\theta_1 + 90°$	20 m	0 m	$90°$
Joint 2	$0°$	$2\text{ m} \leq d_2 \leq 15\text{ m}$	0 m	$90°$
Joint 3	$0°$	$0\text{ m} \leq d_3 \leq 20\text{ m}$	0 m	$0°$

DH Transformation Matrices

- Transformation from LCS_{i-1} to LCS_i

$$A_{i-1,i} = R_{z_{i-1}}(\theta_i) \cdot T_{z_{i-1}}(d_i) \cdot T_{x_i}(a_i) \cdot R_{x_i}(\alpha_i) =$$

$$= \begin{bmatrix} \cos\theta_i & -\sin\theta_i \cdot \cos\alpha_i & \sin\theta_i \cdot \sin\alpha_i & a_i \cdot \cos\theta_i \\ \sin\theta_i & \cos\theta_i \cdot \cos\alpha_i & -\cos\theta_i \cdot \sin\alpha_i & a_i \cdot \sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Exercise 3.1: Transformation Matrix of the Crane

	θ	d	a	α
Joint 1	$\boldsymbol{\theta_1 + 90°}$	$\mathbf{20\ m}$	$\mathbf{0\ m}$	$\mathbf{90°}$
Joint 2	$0°$	$2\ \mathrm{m} \leq d_2 \leq 15\ \mathrm{m}$	$0\ \mathrm{m}$	$90°$
Joint 3	$0°$	$0\ \mathrm{m} \leq d_3 \leq 20\ \mathrm{m}$	$0\ \mathrm{m}$	$0°$

$$T_{0,1} = \begin{pmatrix} \cos(\theta_1 + 90°) & 0 & \sin(\theta_1 + 90°) & 0 \\ \sin(\theta_1 + 90°) & 0 & -\cos(\theta_1 + 90°) & 0 \\ 0 & 1 & 0 & 20 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Exercise 3.1: Transformation Matrix of the Crane

	θ	d	a	α
Joint 1	$\theta_1 + 90°$	$20\ \text{m}$	$0\ \text{m}$	$90°$
Joint 2	$\mathbf{0°}$	$\mathbf{2\ m \leq d_2 \leq 15\ m}$	$\mathbf{0\ m}$	$\mathbf{90°}$
Joint 3	$0°$	$0\ \text{m} \leq d_3 \leq 20\ \text{m}$	$0\ \text{m}$	$0°$

$$T_{0,1} = \begin{pmatrix} \cos(\theta_1 + 90°) & 0 & \sin(\theta_1 + 90°) & 0 \\ \sin(\theta_1 + 90°) & 0 & -\cos(\theta_1 + 90°) & 0 \\ 0 & 1 & 0 & 20 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T_{1,2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Exercise 3.1: Transformation Matrix of the Crane

	θ	d	a	α
Joint 1	$\theta_1 + 90°$	$20\ \text{m}$	$0\ \text{m}$	$90°$
Joint 2	$0°$	$2\ \text{m} \leq d_2 \leq 15\ \text{m}$	$0\ \text{m}$	$90°$
Joint 3	**$0°$**	**$0\ \text{m} \leq d_3 \leq 20\ \text{m}$**	**$0\ \text{m}$**	**$0°$**

$$T_{0,1} = \begin{pmatrix} \cos(\theta_1 + 90°) & 0 & \sin(\theta_1 + 90°) & 0 \\ \sin(\theta_1 + 90°) & 0 & -\cos(\theta_1 + 90°) & 0 \\ 0 & 1 & 0 & 20 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T_{1,2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad T_{2,3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Exercise 3.1: Transformation Matrix of the Crane

$$T_{0,3} = T_{0,1} \cdot T_{1,2} \cdot T_{2,3}$$

Exercise 3.1: Transformation Matrix of the Crane

$$T_{0,3} = T_{0,1} \cdot T_{1,2} \cdot T_{2,3}$$

$$= \begin{pmatrix} \cos(\theta_1 + 90°) & \sin(\theta_1 + 90°) & 0 & \sin(\theta_1 + 90°)\, d_2 \\ \sin(\theta_1 + 90°) & -\cos(\theta_1 + 90°) & 0 & -\cos(\theta_1 + 90°)\, d_2 \\ 0 & 0 & -1 & 20 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot T_{2,3}$$

Exercise 3.1: Transformation Matrix of the Crane

$$T_{0,3} = T_{0,1} \cdot T_{1,2} \cdot T_{2,3}$$

$$= \begin{pmatrix} \cos(\theta_1 + 90°) & \sin(\theta_1 + 90°) & 0 & \sin(\theta_1 + 90°)\, d_2 \\ \sin(\theta_1 + 90°) & -\cos(\theta_1 + 90°) & 0 & -\cos(\theta_1 + 90°)\, d_2 \\ 0 & 0 & -1 & 20 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot T_{2,3}$$

$$= \begin{pmatrix} \cos(\theta_1 + 90°) & \sin(\theta_1 + 90°) & 0 & \sin(\theta_1 + 90°)\, d_2 \\ \sin(\theta_1 + 90°) & -\cos(\theta_1 + 90°) & 0 & -\cos(\theta_1 + 90°)\, d_2 \\ 0 & 0 & -1 & -d_3 + 20 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Exercise 3.2: Jacobian Matrix of the End Effector

$$T_{0,3} = \begin{pmatrix} \cos(\theta_1 + 90°) & \sin(\theta_1 + 90°) & 0 & \sin(\theta_1 + 90°)\, d_2 \\ \sin(\theta_1 + 90°) & -\cos(\theta_1 + 90°) & 0 & -\cos(\theta_1 + 90°)\, d_2 \\ 0 & 0 & -1 & -d_3 + 20 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Determine the Jacobian matrix J.

Exercise 3.2: Jacobian Matrix of the End Effector

$$T_{0,3} = \begin{pmatrix} \cos(\theta_1 + 90°) & \sin(\theta_1 + 90°) & 0 & \sin(\theta_1 + 90°)\, d_2 \\ \sin(\theta_1 + 90°) & -\cos(\theta_1 + 90°) & 0 & -\cos(\theta_1 + 90°)\, d_2 \\ 0 & 0 & -1 & -d_3 + 20 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Determine the Jacobian matrix J.
- Each column of the Jacobian matrix corresponds to a joint θ_i of the kinematic chain

$$J = \begin{pmatrix} \frac{\partial f}{\partial \theta_1} & \cdots & \frac{\partial f}{\partial \theta_n} \end{pmatrix} \in \mathbb{R}^{6\times n}$$

Exercise 3.2: Jacobian Matrix of the End Effector

$$T_{0,3} = \begin{pmatrix} \cos(\theta_1 + 90°) & \sin(\theta_1 + 90°) & 0 & \sin(\theta_1 + 90°)\, d_2 \\ \sin(\theta_1 + 90°) & -\cos(\theta_1 + 90°) & 0 & -\cos(\theta_1 + 90°)\, d_2 \\ 0 & 0 & -1 & -d_3 + 20 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Determine the Jacobian matrix J.
- Each column of the Jacobian matrix corresponds to a joint θ_i of the kinematic chain

$$J = \begin{pmatrix} \frac{\partial f}{\partial \theta_1} & \dots & \frac{\partial f}{\partial \theta_n} \end{pmatrix} \in \mathbb{R}^{6 \times n}$$

$$J = \begin{pmatrix} \frac{\partial f}{\partial \theta_1} & \frac{\partial f}{\partial d_2} & \frac{\partial f}{\partial d_3} \end{pmatrix} \in \mathbb{R}^{6 \times 3}$$

Exercise 3.2: Jacobian Matrix of the End Effector

$$f(\theta_1, d_2, d_3) = (x, y, z, \alpha, \beta, \gamma)^T$$

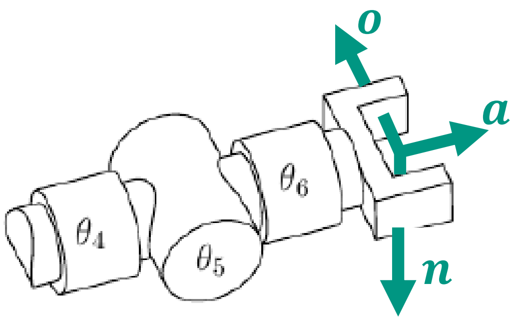
$$T_{0,3} = \begin{pmatrix} \cos(\theta_1 + 90°) & \sin(\theta_1 + 90°) & 0 & \sin(\theta_1 + 90°)\, d_2 \\ \sin(\theta_1 + 90°) & -\cos(\theta_1 + 90°) & 0 & -\cos(\theta_1 + 90°)\, d_2 \\ 0 & 0 & -1 & -d_3 + 20 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

RPY Euler Angles

$$R_s = R_z(\gamma) \cdot R_y(\beta) \cdot R_x(\alpha)$$

$\cos x = cx$
$\sin x = sx$

a: approach
n: normal
o: orientation



$$\begin{pmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{pmatrix} == \begin{pmatrix} c\gamma & -s\gamma & 0 \\ s\gamma & c\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & c\alpha & -s\alpha \\ 0 & s\alpha & c\alpha \end{pmatrix}$$

$$\begin{pmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{pmatrix} = \begin{pmatrix} c\beta \cdot c\gamma & s\alpha \cdot s\beta \cdot c\gamma - c\alpha \cdot s\gamma & s\alpha \cdot s\gamma + c\alpha \cdot s\beta \cdot c\gamma \\ c\beta \cdot s\gamma & s\alpha \cdot s\beta \cdot s\gamma + c\alpha \cdot c\gamma & c\alpha \cdot s\beta \cdot s\gamma - s\alpha \cdot c\gamma \\ -s\beta & s\alpha \cdot c\beta & c\alpha \cdot c\beta \end{pmatrix}$$

Exercise 3.2: Jacobian Matrix of the End Effector

$$f(\theta_1, d_2, d_3) = (x, y, z, \alpha, \beta, \gamma)^T$$

$$T_{0,3} = \begin{pmatrix} \cos(\theta_1 + 90°) & \sin(\theta_1 + 90°) & 0 & \sin(\theta_1 + 90°)\, d_2 \\ \sin(\theta_1 + 90°) & -\cos(\theta_1 + 90°) & 0 & -\cos(\theta_1 + 90°)\, d_2 \\ 0 & 0 & -1 & -d_3 + 20 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$x = \sin(\theta_1 + 90°)\, d_2$$
$$y = -\cos(\theta_1 + 90°)\, d_2$$
$$z = -d_3 + 20$$
$$\beta = \operatorname{asin}(-n_z) = \operatorname{asin}(0) = 0$$

$$\begin{pmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{pmatrix} =$$
$$\begin{pmatrix} c\beta \cdot c\gamma & s\alpha \cdot s\beta \cdot c\gamma - c\alpha \cdot s\gamma & s\alpha \cdot s\gamma + c\alpha \cdot s\beta \cdot c\gamma \\ c\beta \cdot s\gamma & s\alpha \cdot s\beta \cdot s\gamma + c\alpha \cdot c\gamma & c\alpha \cdot s\beta \cdot s\gamma - s\alpha \cdot c\gamma \\ -s\beta & s\alpha \cdot c\beta & c\alpha \cdot c\beta \end{pmatrix}$$

Exercise 3.2: Jacobian Matrix of the End Effector

$$f(\theta_1, d_2, d_3) = (x, y, z, \alpha, \beta, \gamma)^T$$

$$T_{0,3} = \begin{pmatrix} \cos(\theta_1 + 90°) & \sin(\theta_1 + 90°) & 0 & \sin(\theta_1 + 90°)\, d_2 \\ \sin(\theta_1 + 90°) & -\cos(\theta_1 + 90°) & 0 & -\cos(\theta_1 + 90°)\, d_2 \\ 0 & \color{red}{0} & \color{blue}{-1} & -d_3 + 20 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$x = \sin(\theta_1 + 90°)\, d_2$$
$$y = -\cos(\theta_1 + 90°)\, d_2$$
$$z = -d_3 + 20$$
$$\beta = \operatorname{asin}(-n_z) = \operatorname{asin}(0) = 0$$
$$\color{red}{\cos\beta \sin\alpha = 0}, \quad \color{blue}{\cos\beta \cos\alpha = -1}$$
$$\Rightarrow \alpha = \pi$$

$$\begin{pmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{pmatrix} =$$
$$\begin{pmatrix} c\beta \cdot c\gamma & s\alpha \cdot s\beta \cdot c\gamma - c\alpha \cdot s\gamma & s\alpha \cdot s\gamma + c\alpha \cdot s\beta \cdot c\gamma \\ c\beta \cdot s\gamma & s\alpha \cdot s\beta \cdot s\gamma + c\alpha \cdot c\gamma & c\alpha \cdot s\beta \cdot s\gamma - s\alpha \cdot c\gamma \\ -s\beta & \color{red}{s\alpha \cdot c\beta} & \color{blue}{c\alpha \cdot c\beta} \end{pmatrix}$$

Exercise 3.2: Jacobian Matrix of the End Effector

Karlsruhe Institute of Technology

$$f(\theta_1, d_2, d_3) = (x, y, z, \alpha, \beta, \gamma)^T$$

$$T_{0,3} = \begin{pmatrix} \cos(\theta_1 + 90°) & \sin(\theta_1 + 90°) & 0 & \sin(\theta_1 + 90°)\, d_2 \\ \sin(\theta_1 + 90°) & -\cos(\theta_1 + 90°) & 0 & -\cos(\theta_1 + 90°)\, d_2 \\ 0 & 0 & -1 & -d_3 + 20 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$x = \sin(\theta_1 + 90°)\, d_2$$
$$y = -\cos(\theta_1 + 90°)\, d_2$$
$$z = -d_3 + 20$$
$$\beta = \operatorname{asin}(-n_z) = \operatorname{asin}(0) = 0$$
$$\cos\beta \sin\alpha = 0,\ \cos\beta\cos\alpha = -1$$
$$\Rightarrow \alpha = \pi$$
$$\gamma = \operatorname{atan}\left(\frac{n_y}{n_x}\right) = \operatorname{atan}\left(\frac{\sin(\theta_1+90°)}{\cos(\theta_1+90°)}\right)$$

$$\begin{pmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{pmatrix} = \begin{pmatrix} c\beta \cdot c\gamma & s\alpha \cdot s\beta \cdot c\gamma - c\alpha \cdot s\gamma & s\alpha \cdot s\gamma + c\alpha \cdot s\beta \cdot c\gamma \\ c\beta \cdot s\gamma & s\alpha \cdot s\beta \cdot s\gamma + c\alpha \cdot c\gamma & c\alpha \cdot s\beta \cdot s\gamma - s\alpha \cdot c\gamma \\ -s\beta & s\alpha \cdot c\beta & c\alpha \cdot c\beta \end{pmatrix}$$

Exercise 3.2: Jacobian Matrix of the End Effector

$$f(\theta_1, d_2, d_3) = (x, y, z, \alpha, \beta, \gamma)^T$$

$$T_{0,3} = \begin{pmatrix} \cos(\theta_1 + 90°) & \sin(\theta_1 + 90°) & 0 & \sin(\theta_1 + 90°)\, d_2 \\ \sin(\theta_1 + 90°) & -\cos(\theta_1 + 90°) & 0 & -\cos(\theta_1 + 90°)\, d_2 \\ 0 & 0 & -1 & -d_3 + 20 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$x = \sin(\theta_1 + 90°)\, d_2$$
$$y = -\cos(\theta_1 + 90°)\, d_2$$
$$z = -d_3 + 20$$
$$\beta = \operatorname{asin}(-n_z) = \operatorname{asin}(0) = 0$$
$$\cos\beta \sin\alpha = 0,\ \cos\beta\cos\alpha = -1$$
$$\Rightarrow \alpha = \pi$$
$$\gamma = \operatorname{atan}\left(\frac{n_y}{n_x}\right) = \operatorname{atan}\left(\frac{\sin(\theta_1+90°)}{\cos(\theta_1+90°)}\right) = \operatorname{atan}(\tan(\theta_1 + 90°)) = \theta_1 + 90°$$

$$\begin{pmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{pmatrix} = \begin{pmatrix} c\beta \cdot c\gamma & s\alpha \cdot s\beta \cdot c\gamma - c\alpha \cdot s\gamma & s\alpha \cdot s\gamma + c\alpha \cdot s\beta \cdot c\gamma \\ c\beta \cdot s\gamma & s\alpha \cdot s\beta \cdot s\gamma + c\alpha \cdot c\gamma & c\alpha \cdot s\beta \cdot s\gamma - s\alpha \cdot c\gamma \\ -s\beta & s\alpha \cdot c\beta & c\alpha \cdot c\beta \end{pmatrix}$$

Exercise 3.2: Jacobian Matrix of the End Effector

$$f(\theta_1, d_2, d_3) = (x, y, z, \alpha, \beta, \gamma)^T$$

$$x = \sin(\theta_1 + 90°)\, d_2$$
$$y = -\cos(\theta_1 + 90°)\, d_2$$
$$z = -d_3 + 20$$
$$\alpha = \pi$$
$$\beta = 0$$
$$\gamma = \theta_1 + 90°$$

$$J = \begin{pmatrix} \frac{\partial f}{\partial \theta_1} & \frac{\partial f}{\partial d_2} & \frac{\partial f}{\partial d_3} \end{pmatrix} \in \mathbb{R}^{6\times 3}$$

Exercise 3.2: Jacobian Matrix of the End Effector

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$$f(\theta_1, d_2, d_3) = (x, y, z, \alpha, \beta, \gamma)^T$$

$$x = \sin(\theta_1 + 90°)\, d_2$$
$$y = -\cos(\theta_1 + 90°)\, d_2$$
$$z = -d_3 + 20$$
$$\alpha = \pi$$
$$\beta = 0$$
$$\gamma = \theta_1 + 90°$$

$$J = \begin{pmatrix} \frac{\partial f}{\partial \theta_1} & \frac{\partial f}{\partial d_2} & \frac{\partial f}{\partial d_3} \end{pmatrix} \in \mathbb{R}^{6\times 3}$$

$$\frac{\partial f}{\partial \theta_1} = \left(\frac{\partial x}{\partial \theta_1}, \frac{\partial y}{\partial \theta_1} \frac{\partial z}{\partial \theta_1}, \frac{\partial \alpha}{\partial \theta_1}, \frac{\partial \beta}{\partial \theta_1}, \frac{\partial \gamma}{\partial \theta_1} \right)^T$$

H2T

Exercise 3.2: Jacobian Matrix of the End Effector

$$\frac{\partial}{\partial \theta_1}(x) = \frac{\partial}{\partial \theta_1}(\sin(\theta_1 + 90°)\, d_2)$$

$$\frac{\partial}{\partial \theta_1}(y) = \frac{\partial}{\partial \theta_1}(-\cos(\theta_1 + 90°)\, d_2)$$

$$\frac{\partial}{\partial \theta_1}(z) = \frac{\partial}{\partial \theta_1}(-d_3 + 20)$$

Exercise 3.2: Jacobian Matrix of the End Effector

$$\frac{\partial}{\partial \theta_1}(x) = \frac{\partial}{\partial \theta_1}(\sin(\theta_1 + 90°)\, d_2)$$

$$= \cos(\theta_1 + 90°)\, d_2$$

$$\frac{\partial}{\partial \theta_1}(y) = \frac{\partial}{\partial \theta_1}(-\cos(\theta_1 + 90°)\, d_2)$$

$$= \sin(\theta_1 + 90°)\, d_2$$

$$\frac{\partial}{\partial \theta_1}(z) = \frac{\partial}{\partial \theta_1}(-d_3 + 20) = 0$$

Exercise 3.2: Jacobian Matrix of the End Effector

$$\frac{\partial}{\partial \theta_1}(\alpha) = \frac{\partial}{\partial \theta_1}(\pi)$$

$$\frac{\partial}{\partial \theta_1}(\beta) = \frac{\partial}{\partial \theta_1}(0)$$

$$\frac{\partial}{\partial \theta_1}(\gamma) = \frac{\partial}{\partial \theta_1}(\theta_1 + 90°)$$

$$\frac{\partial f}{\partial \theta_1} =$$

Exercise 3.2: Jacobian Matrix of the End Effector

Karlsruhe Institute of Technology

$$\frac{\partial}{\partial \theta_1}(\alpha) = \frac{\partial}{\partial \theta_1}(\pi) = 0$$

$$\frac{\partial}{\partial \theta_1}(\beta) = \frac{\partial}{\partial \theta_1}(0) = 0$$

$$\frac{\partial}{\partial \theta_1}(\gamma) = \frac{\partial}{\partial \theta_1}(\theta_1 + 90°) = 1$$

$$\frac{\partial f}{\partial \theta_1} = (\cos(\theta_1 + 90°)\, d_2, \sin(\theta_1 + 90°)\, d_2, 0, 0, 0, 1)^T$$

Exercise 3.2: Jacobian Matrix of the End Effector

$$\frac{\partial}{\partial d_2}(x) = \frac{\partial}{\partial d_2}(\sin(\theta_1 + 90°)\, d_2)$$

$$\frac{\partial}{\partial d_2}(y) = \frac{\partial}{\partial d_2}(-\cos(\theta_1 + 90°)\, d_2)$$

$$\frac{\partial}{\partial d_2}(z) = \frac{\partial}{\partial d_2}(\alpha) = \frac{\partial}{\partial d_2}(\beta) = \frac{\partial}{\partial d_2}(\gamma)$$

Exercise 3.2: Jacobian Matrix of the End Effector

$$\frac{\partial}{\partial d_2}(x) = \frac{\partial}{\partial d_2}(\sin(\theta_1 + 90°)\, d_2)$$
$$= \sin(\theta_1 + 90°)$$

$$\frac{\partial}{\partial d_2}(y) = \frac{\partial}{\partial d_2}(-\cos(\theta_1 + 90°)\, d_2)$$
$$= -\cos(\theta_1 + 90°)$$

$$\frac{\partial}{\partial d_2}(z) = \frac{\partial}{\partial d_2}(\alpha) = \frac{\partial}{\partial d_2}(\beta) = \frac{\partial}{\partial d_2}(\gamma) = 0$$

Exercise 3.2: Jacobian Matrix of the End Effector

$$\frac{\partial}{\partial d_3}(z) = \frac{\partial}{\partial d_3}(-d_3 + 20)$$

$$\frac{\partial}{\partial d_3}(x) = \frac{\partial}{\partial d_3}(y) = \frac{\partial}{\partial d_3}(\alpha) = \frac{\partial}{\partial d_3}(\beta) = \frac{\partial}{\partial d_3}(\gamma)$$

Exercise 3.2: Jacobian Matrix of the End Effector

$$\frac{\partial}{\partial d_3}(z) = \frac{\partial}{\partial d_3}(-d_3 + 20)$$

$$= -1$$

$$\frac{\partial}{\partial d_3}(x) = \frac{\partial}{\partial d_3}(y) = \frac{\partial}{\partial d_3}(\alpha) = \frac{\partial}{\partial d_3}(\beta) = \frac{\partial}{\partial d_3}(\gamma) = 0$$

Exercise 3.2: Jacobian Matrix of the End Effector

$$J = \begin{pmatrix} \frac{\partial f}{\partial \theta_1} & \frac{\partial f}{\partial d_2} & \frac{\partial f}{\partial d_3} \end{pmatrix} \in \mathbb{R}^{6\times 3}$$

Exercise 3.2: Jacobian Matrix of the End Effector

$$J = \begin{pmatrix} \frac{\partial f}{\partial \theta_1} & \frac{\partial f}{\partial d_2} & \frac{\partial f}{\partial d_3} \end{pmatrix} \in \mathbb{R}^{6\times 3}$$

$$J = \begin{pmatrix} \cos(\theta_1 + 90°)\, d_2 & \sin(\theta_1 + 90°) & 0 \\ \sin(\theta_1 + 90°)\, d_2 & -\cos(\theta_1 + 90°) & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Exercise 3.3: Velocity of the End Effector (1,1)

- Determine the end effector velocity resulting from the following combination of the crane configuration $\boldsymbol{q}$ and the joint velocity $\boldsymbol{p}$.

$$\boldsymbol{q}_1 = (90, 10, 10)^T, \boldsymbol{p}_1 = (1, 1, 1)^T$$

Exercise 3.3: Velocity of the End Effector (1,1)

- Determine the end effector velocity resulting from the following combination of the crane configuration $\boldsymbol{q}$ and the joint velocity $\boldsymbol{p}$.

$$\boldsymbol{q}_1 = (90, 10, 10)^T, \boldsymbol{p}_1 = (1, 1, 1)^T$$

$$\boldsymbol{v}_1 = J(\boldsymbol{q}_1) \cdot \boldsymbol{p}_1$$

Exercise 3.3: Velocity of the End Effector (1,1)

- Determine the end effector velocity resulting from the following combination of the crane configuration $\boldsymbol{q}$ and the joint velocity $\boldsymbol{p}$.

$$\boldsymbol{q}_1 = (90, 10, 10)^T, \boldsymbol{p}_1 = (1, 1, 1)^T$$

$$\boldsymbol{v}_1 = J(\boldsymbol{q}_1) \cdot \boldsymbol{p}_1$$

$$\boldsymbol{v}_1 = \begin{pmatrix} \cos(\theta_1 + 90°) \cdot d_2 & \sin(\theta_1 + 90°) & 0 \\ \sin(\theta_1 + 90°) \cdot d_2 & -\cos(\theta_1 + 90°) & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Exercise 3.3: Velocity of the End Effector (1,1)

- Determine the end effector velocity resulting from the following combination of the crane configuration $\boldsymbol{q}$ and the joint velocity $\boldsymbol{p}$.

$$\boldsymbol{q}_1 = (90, 10, 10)^T, \boldsymbol{p}_1 = (1, 1, 1)^T$$

$$\boldsymbol{v}_1 = J(\boldsymbol{q}_1) \cdot \boldsymbol{p}_1$$

$$\boldsymbol{v}_1 = \begin{pmatrix} \cos(90° + 90°) \cdot 10 & \sin(90° + 90°) & 0 \\ \sin(90° + 90°) \cdot 10 & -\cos(90° + 90°) & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Exercise 3.3: Velocity of the End Effector (1,1)

- Determine the end effector velocity resulting from the following combination of the crane configuration $\boldsymbol{q}$ and the joint velocity $\boldsymbol{p}$.

$$\boldsymbol{q}_1 = (90, 10, 10)^T, \boldsymbol{p}_1 = (1, 1, 1)^T$$

$$\boldsymbol{v}_1 = J(\boldsymbol{q}_1) \cdot \boldsymbol{p}_1$$

$$\boldsymbol{v}_1 = \begin{pmatrix} -10 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Exercise 3.3: Velocity of the End Effector (1,1)

- Determine the end effector velocity resulting from the following combination of the crane configuration $\boldsymbol{q}$ and the joint velocity $\boldsymbol{p}$.

$$\boldsymbol{q}_1 = (90, 10, 10)^T, \boldsymbol{p}_1 = (1, 1, 1)^T$$

$$\boldsymbol{v}_1 = J(\boldsymbol{q}_1) \cdot \boldsymbol{p}_1$$

$$\boldsymbol{v}_1 = \begin{pmatrix} -10 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -10 \\ 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Exercise 3.3: Velocity of the End Effector (1,2)

- Determine the end effector velocity resulting from the following combination of the crane configuration $\boldsymbol{q}$ and the joint velocity $\boldsymbol{p}$.

$$\boldsymbol{q}_1 = (90, 10, 10)^T, \boldsymbol{p}_2 = (-1, -1, 0)^T$$

$$\boldsymbol{v}_2 = J(\boldsymbol{q}_1) \cdot \boldsymbol{p}_2$$

Exercise 3.3: Velocity of the End Effector (1,2)

- Determine the end effector velocity resulting from the following combination of the crane configuration $\boldsymbol{q}$ and the joint velocity $\boldsymbol{p}$.

$$\boldsymbol{q}_1 = (90, 10, 10)^T, \boldsymbol{p}_2 = (-1, -1, 0)^T$$

$$\boldsymbol{v}_2 = J(\boldsymbol{q}_1) \cdot \boldsymbol{p}_2$$

$$\boldsymbol{v}_2 = \begin{pmatrix} -10 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$$

Exercise 3.3: Velocity of the End Effector (1,2)

- Determine the end effector velocity resulting from the following combination of the crane configuration $\boldsymbol{q}$ and the joint velocity $\boldsymbol{p}$.

$$\boldsymbol{q}_1 = (90, 10, 10)^T, \boldsymbol{p}_2 = (-1, -1, 0)^T$$

$$\boldsymbol{v}_2 = J(\boldsymbol{q}_1) \cdot \boldsymbol{p}_2$$

$$\boldsymbol{v}_2 = \begin{pmatrix} -10 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 10 \\ -1 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

Exercise 3.3: Velocity of the End Effector (2,2)

- Determine the end effector velocity resulting from the following combination of the crane configuration $\boldsymbol{q}$ and the joint velocity $\boldsymbol{p}$.

$$\boldsymbol{q}_2 = (180, 2, 15)^T, \boldsymbol{p}_2 = (-1, -1, 0)$$

$$\boldsymbol{v}_3 = J(\boldsymbol{q}_2) \cdot \boldsymbol{p}_2$$

Exercise 3.3: Velocity of the End Effector (2,2)

- Determine the end effector velocity resulting from the following combination of the crane configuration $\boldsymbol{q}$ and the joint velocity $\boldsymbol{p}$.

$$\boldsymbol{q}_2 = (180, 2, 15)^T, \boldsymbol{p}_2 = (-1, -1, 0)$$

$$\boldsymbol{v}_3 = J(\boldsymbol{q}_2) \cdot \boldsymbol{p}_2$$

$$\boldsymbol{v}_3 = \begin{pmatrix} \cos(\theta_1 + 90°) \cdot d_2 & \sin(\theta_1 + 90°) & 0 \\ \sin(\theta_1 + 90°) \cdot d_2 & -\cos(\theta_1 + 90°) & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$$

Exercise 3.3: Velocity of the End Effector (2,2)

- Determine the end effector velocity resulting from the following combination of the crane configuration $\boldsymbol{q}$ and the joint velocity $\boldsymbol{p}$.

$$\boldsymbol{q}_2 = (180, 2, 15)^T, \boldsymbol{p}_2 = (-1, -1, 0)$$

$$\boldsymbol{v}_3 = J(\boldsymbol{q}_2) \cdot \boldsymbol{p}_2$$

$$\boldsymbol{v}_3 = \begin{pmatrix} \cos(180° + 90°) \cdot 2 & \sin(180° + 90°) & 0 \\ \sin(180° + 90°) \cdot 2 & -\cos(180° + 90°) & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$$

Exercise 3.3: Velocity of the End Effector (2,2)

- Determine the end effector velocity resulting from the following combination of the crane configuration $\boldsymbol{q}$ and the joint velocity $\boldsymbol{p}$.

$$\boldsymbol{q}_2 = (180, 2, 15)^T, \boldsymbol{p}_2 = (-1, -1, 0)$$

$$\boldsymbol{v}_3 = J(\boldsymbol{q}_2) \cdot \boldsymbol{p}_2$$

$$\boldsymbol{v}_3 = \begin{pmatrix} 0 & -1 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$$

Exercise 3.3: Velocity of the End Effector (2,2)

- Determine the end effector velocity resulting from the following combination of the crane configuration $\boldsymbol{q}$ and the joint velocity $\boldsymbol{p}$.

$$\boldsymbol{q}_2 = (180, 2, 15)^T, \boldsymbol{p}_2 = (-1, -1, 0)$$

$$\boldsymbol{v}_3 = J(\boldsymbol{q}_2) \cdot \boldsymbol{p}_2$$

$$\boldsymbol{v}_3 = \begin{pmatrix} 0 & -1 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

Exercise 3.3: Velocity of the End Effector (2,3)

- Determine the end effector velocity resulting from the following combination of the crane configuration $\boldsymbol{q}$ and the joint velocity $\boldsymbol{p}$.

$$\boldsymbol{q}_2 = (180, 2, 15)^T, \boldsymbol{p}_3 = (2, -1, 2)$$

$$\boldsymbol{v}_4 = J(\boldsymbol{q}_2) \cdot \boldsymbol{p}_3$$

Exercise 3.3: Velocity of the End Effector (2,3)

- Determine the end effector velocity resulting from the following combination of the crane configuration $\boldsymbol{q}$ and the joint velocity $\boldsymbol{p}$.

$$\boldsymbol{q}_2 = (180, 2, 15)^T, \boldsymbol{p}_3 = (2, -1, 2)$$

$$\boldsymbol{v}_4 = J(\boldsymbol{q}_2) \cdot \boldsymbol{p}_3$$

$$\boldsymbol{v}_4 = \begin{pmatrix} 0 & -1 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$$

Exercise 3.3: Velocity of the End Effector (2,3)

- Determine the end effector velocity resulting from the following combination of the crane configuration $\boldsymbol{q}$ and the joint velocity $\boldsymbol{p}$.

$$\boldsymbol{q}_2 = (180, 2, 15)^T, \boldsymbol{p}_3 = (2, -1, 2)$$

$$\boldsymbol{v}_4 = J(\boldsymbol{q}_2) \cdot \boldsymbol{p}_3$$

$$\boldsymbol{v}_4 = \begin{pmatrix} 0 & -1 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \\ -2 \\ 0 \\ 0 \\ 2 \end{pmatrix}$$